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Homomorphism on Fuzzy ℓ - Ideals

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***Abstract*— this paper pursues an investigation on fuzzy ℓ-ideals. We studied the concept of homomorphism on fuzzy ℓ-ideals and we obtain some related results.**

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I. INTRODUCTION

The concept of a fuzzy set was first introduced by Zadeh[14] and this concept was adapted by Goguen [15] to study fuzzy relations. Rosenfeld applied it to group theory and developed the theory of fuzzy groups. Since then researchers in various discipline of Mathematics have been trying to extend their ideas to the broader frame work of the fuzzy setting and the concepts of fuzzy lattice, fuzzy ideal, fuzzy prime ideals in lattice were introduced by many authors. U.M.Swamy and Viswananda Raju [13] developed the theory of fuzzy ℓ-ideals and gave some interesting results. The partially ordered algebraic systems play an important role in algebra. Some important concepts in partially ordered systems are lattice ordered groups and lattice ordered rings. These concepts play a major role in many branches of Algebra. This provides a sufficient motivation to researchers to review various concepts and results of partially ordered algebraic systems with the frame work of fuzzy setting. G.S.V.SathyaSaiBaba [12] studied fuzzy lattice ordered groups as a mapping from lattice ordered group into a complete lattice. He introduced L-fuzzy ℓ-ideals of fuzzy ℓ-group and developed the theory of fuzzy ℓ-ideals by proving the existence of one to one correspondence between the Lattice of all *L*-fuzzy ℓ - ideals and the lattice of all *L*-fuzzy congruence of an *l*-group *G*. (Lattice Isomorphism). In this paper, the concept of homomorphism of fuzzy ℓ-ideal is studied and some results are established.

II. PRELIMINARIES

Definition 2.1 A non-empty set G is called a lattice ordered group (ℓ - group) iff

(i) $(G,+)$ is a group

(ii) $(G,\leq)$ is a lattice

(iii) $x \leq y$ implies $a+x+b \leq a+y+b$ for all $a,b,x,y \in G$.

Definition 2.2 A non-empty set G is called a commutative lattice ordered group (ℓ - group) iff
(i) $(G,+)$ is a group

(ii) $(G,\vee,\wedge)$ is a lattice.

(iii) $a+(x\vee y)=(a+x)\vee(a+y)$ and

$a+(x\wedge y)=(a+x)\wedge(a+y)$ for all $a,b,x,y \in G$.

Result 2.3 The above two definitions of ℓ - group are equivalent.
Definition 2.4 Let G be a ℓ - group. A non-empty subset I of G is called an ℓ -ideal of G if

(i) I is a subgroup of G.

(ii) I is a sub lattice of G.

(iii) $0 < x < a$ and $a \in I \Rightarrow x \in 1$

Definition 2.5 A fuzzy set is a pair (X,μ), where X is any non empty set and $\mu : X \to [0,1]$.

Definition 2.6 Let μ be a fuzzy set on a non empty set X and $t \in [0,1]$. Then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called the level set of μ.

Definition 2.7 let μ be a fuzzy set on a non empty set X. Then the set $\{x \in X / \mu(x) > 0\}$ is called the support of μ and it is denoted by Supp (μ).

III. FUZZY ℓ-IDEALS IN ℓ-GROUP

Definition 3.1 Let G be a commutative ℓ-group. A fuzzy set μ of G is said to be fuzzy ℓ-ideal of G if

(i) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$

(ii) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$

(iii) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$

(iv) $0 < x < a \Rightarrow \mu(x) \geq \mu(a)$ for all $x, y, a, b \in G$.

Definition 3.2 The union of two fuzzy ℓ-ideals μ_1 and μ_2 of a commutative ℓ-group G denoted by $(\mu_1 \cup \mu_2)$ is a fuzzy subset of G defined by $(\mu_1 \cup \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\}$ for all $x \in G$.

The intersection of two fuzzy ℓ-ideals μ_1 and μ_2 of a commutative ℓ-group G denoted by $(\mu_1 \cap \mu_2)$ is a fuzzy subset of G defined by $(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\}$ for all $x \in G$.

Definition 3.3 Let μ_1 and μ_2 be any two fuzzy ℓ-ideals of a commutative ℓ-group G. Then μ_1 is said to be contained in μ_2 denoted by $\mu_1 \subseteq \mu_2$ if $\mu_1(x) \leq \mu_2(x)$ for all $x \in G$. If $\mu_1(x) = \mu_2(x)$ for all $x \in G$ then μ_1 and μ_2 are said to be equal and we can write $\mu_1 = \mu_2$.

Proposition 3.4 Let μ_1 and μ_2 be any two fuzzy ℓ-ideals of a commutative ℓ-group G. If $\mu_1 \subseteq \mu_2$ then $\mu_1 \cup \mu_2 = \mu_2$ and $\mu_1 \cap \mu_2 = \mu_1$.

Proposition 3.5 Let μ_1 and μ_2 be any two fuzzy ℓ-ideals of an commutative ℓ-group G. Then $\mu_1 \cup \mu_2 \supseteq \mu_1 \cap \mu_2$.

Proposition 3.6 Intersection of any two fuzzy ℓ-ideals of an commutative ℓ-group is a fuzzy ℓ-ideal.

Definition 3.7 **If** μ_1 and μ_2 are any two fuzzy ℓ-ideals of a commutative ℓ-group G the join of μ_1 and μ_2 defined by

$(\mu_1 \vee \mu_2)(x) = \sup_{x = y \vee z} \{\min\{\mu_1(y), \mu_2(z)\}\}$, where $x, y, z \in G$.

The meet of μ_1 and μ_2 defined by

$(\mu_1 \wedge \mu_2)(x) = \sup_{x = y \wedge z} \{\min\{\mu_1(y), \mu_2(z)\}\}$, where $x, y, z \in G$.

Proposition 3.8 If μ_1 and μ_2 are any two fuzzy ℓ-ideals of a commutative ℓ-group G then $\mu_1 \wedge \mu_2 = \mu_1 \cap \mu_2$.

Proposition 3.9 If μ_1 and μ_2 are any two fuzzy ℓ-ideals of a commutative ℓ-group G then $\mu_1 \wedge \mu_2$ is also a fuzzy ℓ-ideal of G**.**

Proposition 3.10 Let G be a commutative ℓ-group. If μ is a fuzzy ℓ-ideal of G then $Supp(\mu)$ is an ℓ-ideal of G if $Supp \neq \phi$.

Proposition 3.11 If μ_1 is any fuzzy ℓ-ideal of an commutative ℓ-group G then $\mu(1) \leq \mu(x) \leq \mu(0)$ for all $x \in G$ where 0 is the least element and 1 is the greatest element in G.

Proposition 3.12 Every constant function of a commutative ℓ-group G is a fuzzy ℓ-ideal of G.

Proposition 3.13 (*Characterization Theorem for fuzzy ℓ-ideals)*

Let G be a commutative ℓ - group. A fuzzy set μ of G is a fuzzy ℓ -ideal of G if and only if the set $\mu_t = \{x \in G / \mu(x) \geq t\}$ is an ℓ -ideal of G for all $t \in [0,1]$ with $\mu_t \neq \phi$.

Definition 3.14 Let μ be a fuzzy ℓ-ideal of a commutative ℓ-group G and $t \leq \mu(0)$. Then the ℓ-ideal μ_t of G is called a level ℓ-ideal of μ.

Proposition 3.15 Let μ be a fuzzy ℓ-ideal of a commutative ℓ-group G. If $\mu(x-y) = \mu(0) = \mu(0)$ for some *x,y* in *G* then $\mu(x) = \mu(y)$.

Proposition 3.16 Let μ be a fuzzy ℓ-ideal of a commutative ℓ-group G. If $\mu(x) < \mu(y)$ for some *x,y* in *G* then $\mu(x-y) = \mu(x)$.

IV. HOMOMORPHISM ON FUZZY ℓ-IDEALS

Definition 4.1 Let G be a lattice ordered group and $\mu : G \to [0,1]$ be the fuzzy ℓ-ideal. Then the fuzzy set $x+\mu : G \to [0,1]$ defined by $(x+\mu)(y) = \mu(y-x)$ is called a coset of the fuzzy ℓ-ideal μ. The set of all cosets of μ in G is denoted by G_μ.

Proposition 4.2 $\mathbf{G_\mu}$ is a lattice ordered group under the following operations:

- $(x+\mu)\vee(y+\mu) = (x\vee y)+\mu$
- $(x+\mu)\wedge(y+\mu) = (x\wedge y)+\mu$
- $(x+\mu)+(y+\mu) = (x+y)+\mu$

Proposition 4.3 Let μ is a fuzzy ℓ-ideal of a commutative ℓ-group G. Then $\mu(x) = \mu(0)$ if and only if $x+\mu = 0+\mu$.

Proof

Assume that $\mu(x) = \mu(0)$

$\Rightarrow \mu(r) \le \mu(0) = \mu(x)$

$\Rightarrow \mu(r) \le \mu(x)$

$\Rightarrow \mu(r-x) = \mu(r)$

Now $(0+\mu)r = \mu(r-0) = \mu(r)$

$= \mu(r-x)$

$= (x+\mu)r$

$\Rightarrow x+\mu = 0+\mu$

Conversely assume that $0+\mu = x+\mu$

$\Rightarrow (0+\mu)r = (r+\mu)$ for some r in G.

$\Rightarrow \mu(r-0) = \mu(r-r)$

$\Rightarrow \mu(r) = \mu(0) \ldots .(1)$

Suppose $x+\mu = 0+\mu$

$(x+\mu)r = (0+\mu)r$

$\Rightarrow \mu(r-x) = \mu(r-0) = \mu(r) \ldots .(2)$

Form (1) and (2) we get $\mu(r-x) = \mu(0)$

$\Rightarrow \mu(x) = \mu(r) = \mu(0)$

$\Rightarrow \mu(x) = \mu(0)$

Proposition 4.4 **Let** G and G' be two lattice ordered groups. $f : G \to G'$ be any function and μ_1 be the fuzzy ℓ-ideal of G, μ_2 be the fuzzy ℓ-ideal of G'. Define the pre-image of f as $f^{-1}[(\mu_2)] = \mu_2[f(x)]$ for all x in G. Then $f^{-1}(\mu_2)$ is a fuzzy ℓ-ideal of G'.

Proof:

(i) $f^{-1}(\mu_2(x\vee y)) = \mu_2[f(x\vee y)]$

$= \mu_2[f(x)\vee f(y)]$

$\ge \mu_2[f(x)] \wedge \mu_2[f(y)]$

$= f^{-1}[\mu_2(x)] \wedge f^{-1}[\mu_2(y)]$

$\Rightarrow f^{-1}(\mu_2(x\vee y)) \ge f^{-1}[\mu_2(x)] \wedge f^{-1}[\mu_2(y)]$

(ii) $f^{-1}(\mu_2(x\wedge y)) = \mu_2[f(x\wedge y)]$

$= \mu_2[f(x)\wedge f(y)]$

$\ge \mu_2[f(x)] \wedge \mu_2[f(y)]$

$= f^{-1}[\mu_2(x)] \wedge f^{-1}[\mu_2(y)]$

$\Rightarrow f^{-1}(\mu_2(x\wedge y)) \ge f^{-1}[\mu_2(x)] \wedge f^{-1}[\mu_2(y)]$

(iii) $f^{-1}(\mu_2(x+y)) = \mu_2[f(x+y)]$

$= \mu_2[f(x)+f(y)]$

$\ge \mu_2[f(x)] + \mu_2[f(y)]$

$= f^{-1}[\mu_2(x)] + f^{-1}[\mu_2(y)]$

$\Rightarrow f^{-1}(\mu_2(x+y)) \ge f^{-1}[\mu_2(x)] + f^{-1}[\mu_2(y)]$

Let $0 < x < a$ in G'

Since μ_2 is the fuzzy ℓ-ideal of G', we have $\mu_2(x) \ge \mu_2(a)$

$\Rightarrow f^{-1}[\mu_2(x)] = \mu_2[f(x)] \ge \mu_2[f(a)] = f^{-1}[\mu_2(a)]$

$\Rightarrow f^{-1}[\mu_2(x)] \ge f^{-1}[\mu_2(a)]$

Hence $f^{-1}(a)$ is a fuzzy ℓ-ideal of G'.

Proposition 4.5 *[Fundamental theorem of Homomorphism]*

Let μ be any fuzzy ℓ-ideal of the lattice ordered group *G*.

Then $G/\mu_t \cong G_\mu$ where $t = \mu(0)$.

Proof

Let μ be a fuzzy ℓ-ideal of G and G_μ be the set of all cosets of μ in G.

Define $f : G \to G_\mu$ by $f(x) = x + \mu$

(i) f is well defined

Let $x = y$

$\Rightarrow x + \mu = y + \mu$

$\Rightarrow f(x) = f(y)$

(ii) f is onto

Let $x + \mu \in G_\mu$.

Then there exists x in G such that

$f(x) = x + \mu$.

(iii) f preserves ∧ , ∨ and +.

$$f(x+y) = (x+y)+\mu$$
$$= (x+\mu)+(y+\mu)$$
$$= f(x)+f(y)$$
$$f(x\vee y) = (x\vee y)+\mu$$
$$= (x+\mu)\vee(y+\mu)$$
$$= f(x)\vee f(y)$$
$$f(x\wedge y) = (x\wedge y)+\mu$$
$$= (x+\mu)\wedge(y+\mu)$$
$$= f(x)\wedge f(y)$$

Suppose $x \in \ker f$ *iff* $x + \mu = 0 + \mu$

$\Rightarrow \mu(x) = \mu(0) = t$

$\Rightarrow x \in \mu_t$

$\Rightarrow \ker f = \mu_t$

Hence $G\mu_t \cong G_\mu$.

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