IFERP
connecting engineers... developing research

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)

Vol 8, Issue 9, September 2021

Modified Implicit Method for Solving One Dimensional Heat Equation

Nabila F. Kaskar

Department of Mathematics, University of Mumbai, Maharashtra, India

***Abstract*----An explicit method is computationally simple but it is only valid for 0<k/h^2 =r≤0.5. Crank and Nicolson (1947) proposed and used an implicit method that is valid for all finite values of r. We proposed a modified Implicit method for solving one dimensional heat equation with initial and boundary conditions and this method is valid for 0<r≤1.0. Test examples solved using modified implicit method gave good approximations to exact solutions of parabolic partial differential equations. We also observed that percentage errors computed in test examples which are solved by our method are less than those percentage errors in respective test examples solved by crank Nicolson Implicit method.**

***Index Terms*—Crank-Nicolson Method, Explicit Method, Implicit Method**

I. INTRODUCTION

One Dimensional parabolic partial differential equation $\frac{\partial u}{\partial t} = \frac{k\partial^2 u}{\partial x^2}$ derives from the theory of heat Conduction is simplest and are well-Known equations, this is determined by the passage of time t and one or more spatial factors. Fluid flow, electrostatics, electrodynamics, elasticity, and other practical applications rely heavily on these equations. In one-dimensional space, L Euler (1707-1783) developed the finite difference approximation, which was later modified to two-dimensional space by C Runge. A new approach to the finite difference method in numerical applications was developed in the early 1950s. The theoretical findings of the finite difference approach for solving PDEs were obtained in terms of accuracy, stability, and convergence. PDEs are converted to a system of linear equations that can be solved rapidly and easily using the matrix algebra technique utilising the Finite Difference Method (FDM). Analytically solving parabolic PDEs is challenging; only a small number of problems can be solved this way, and the utility of the solution is even lower in situations requiring shapes that satisfy Boundary Conditions. Numerical approaches are one of the few possibilities for tackling the problem in such cases. There are various different Numerical Methods that can be used to get a good approximate result. According to Smith, the most often utilised method for solving differential equations is FDM. Depending on the finite difference approximation used for a specific differential equation, there are a variety of finite difference approaches, including explicit and implicit methods. Du Fort and Frankel improved the simple explicit approach in 1953, and it proved to be more stable than the simple explicit example. When solving partial differential equations with the finite difference method, you can compare error analysis both analytically and numerically.

John Crank and Phyllis developed the Crank Nicolson forward time central space for solving parabolic PDEs in 1947. Because there was no information for negative t at the start, the time derivative was substituted by forward difference in time, according to Kreyszing in 1993. Smaller values of r Yields more accurate results, so Crank Nicolson method was chosen for study. Due to Unconditional stability no matter how small r becomes, Computer implementation was always easy.

In this paper we proposed Modified Implicit method for solving parabolic heat partial differential equation and compare the results obtained with the exact solution. There are many exhaustive text in [2], [3], [4], and [7] just to mention few.

II. FINITE DIFFERENCE METHOD

In general, the finite difference method yields answers that are either as precise as the data permits or as precise as the technical goals need. In both cases, a finite difference approach is as good as one derived using an analytical formula. It is worth noting that changing the beginning and boundary conditions of partial differential equations frequently makes analytical solutions impossible, yet these changes have no effect on the finite difference approach.

At the locations of intersection of parallel lines, known as mesh points or nodal points, the numerical values of the dependent variables are obtained. Because the values in the finite difference technique are allocated solely to the

IFERP
connecting engineers... developing research

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)

Vol 8, Issue 9, September 2021

node points, we cannot determine the values between the node points, which is the finite difference method's restriction.

III. FINITE DIFFERENCE APPROXIMATION TO PARTIAL DERIVATIVES

Assume u is a function of the independent variables x and y. Subdivide xy-plane into sets of equal rectangles of sides h, k by equally spaced grid lines parallel to Oy, defined by $x_i = ih,\ i = 0, \pm1, \pm2, \pm3,$ ...and equally spaced grid lines parallel to Ox, defined by $y_j = jk, j = 0,1,2,\ldots$

Denote the value u at the representative mesh point

$\frac{\partial u}{\partial x} = \frac{u_{i+1,j}-u_{i,j}}{h} + o(h)$. This is forward difference approximation.

$\frac{\partial u}{\partial x} = \frac{u_{i+1,j}-u_{i-1,j}}{2h} + o(h^2)$. This is central difference approximation.

$\frac{\partial u}{\partial x} = \frac{u_{i,j}-u_{i-1,j}}{h} + o(h)$. This is backward difference approximation.

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + o(h^2).$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2} + O(k^2)$$

The common finite difference method for solving partial differential equation is explicit method, implicit method and Crank Nicolson Method.

IV. ONE DIMENSIONAL HEAT EQUATION

Consider a one dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad . \qquad \textbf{(1)}$$

Under the boundary conditions $u(0,t) = T_0$, $u(l,t) = T_1$ and initial condition $u(x,0) = f(x)$, by substituting u_{xx} and u_t in (1) with finite difference approximations we obtain formula as

$$u_{i,j+1} = r\, u_{i-1,j} + (1-2r)u_{i,j} + r\, u_{i+1,j}. \qquad (2)$$

where $r = \frac{k}{h^2}$. This formula gives the unknown temperature $u_{i,j+1}$ at the $(i, j+1)$ th mesh point in terms of known temperature along the j th time row. Thus equation (2) presents explicit formula. This explicit method is valid only for $0 < r \le \frac{1}{2}$.

A. *Crank-Nicolson Implicit Method*

Crank and Nicolson proposed a method by considering the partial differential equation as being satisfied at the midpoint of $\{ ih, (j+\frac{1}{2})k\}$ and replace $\frac{\partial^2 u}{\partial x^2}$ by the mean of its finite difference approximations at j th and $(j+1)$th level in $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ and obtain the formula.

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i+1,j} + (2-2r)u_{i,j} + ru_{i-1,j} \qquad (3)$$

Equation (3) is Crank Nicolson's difference formula.

In general, three unknown and three known pivotal values of are found on the left side of equation (3) u.

V. MODIFIED IMPLICIT METHOD FOR ONE DIMENSIONAL HEAT EQUATION

Modified approach for solving parabolic heat equation with respect to corresponding initial and boundary conditions. Replacing u_{xx} and u_t in (1) with respective finite difference approximation i.e.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 x}{\partial x^2}$$

$$\therefore\ \frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{1}{2}\left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right]$$

$$\therefore\ \frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{1}{2h^2}\left[u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Where $r = \frac{k}{h^2}$

$$\therefore\ u_{i,j+1} - u_{i,j-1} = ru_{i-1,j+1} - 2ru_{i,j+1} + ru_{i+1,j+1} + ru_{i-1,j} - 2ru_{i,j} + ru_{i+1,j}$$

$$\therefore\ u_{i,j+1} = ru_{i-1,j+1} - 2ru_{i,j+1} + ru_{i+1,j+1} + ru_{i-1,j} - 2ru_{i,j} + ru_{i+1,j} + u_{i,j-1}$$

By considering average of $j+1$ and $j-1$ level as $u_{i,j} = \frac{u_{i,j+1}+u_{i,j-1}}{2}$

$$\therefore u_{i,j+1} = ru_{i-1,j+1} - 2ru_{i,j+1} + ru_{i+1,j+1} + ru_{i-1,j} - 2r\left[\frac{u_{i,j+1}+u_{i,j-1}}{2}\right] + ru_{i+1,j} + u_{i,j-1}$$

$$\therefore u_{i,j+1} = ru_{i-1,j+1} - 2ru_{i,j+1} + ru_{i+1,j+1} + ru_{i-1,j} - ru_{i,j+1} - ru_{i,j-1} + ru_{i+1,j} + u_{i,j-1}$$

$$\therefore -ru_{i-1,j+1} + (1+3r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + ru_{i+1,j} + (1-r)u_{i,j-1}. \quad (4)$$

Equation (4) is our proposed Modified Implicit method for solving heat equation. The first level t_1 is calculated by equation (3).

The Modified Implicit method is demonstrated by calculating Numerical solution of following Examples

Test Example-1: Solve the partial differential heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$ with initial condition $u(x,0) = x(4-x)$ and Boundary condition $u(0,t) = 0$, $u(4,t) = 0$ for $t > 0$

Solution:

Analytic solution of this partial differential equation is

$$u(x,t) = \sum \frac{32}{n^3\pi^3}[1-\cos(n\pi)]e^{\frac{-n^2\pi^2 t}{9}}\sin(\frac{n\pi x}{3})$$

$t\backslash x$	$x = 0.4$	$x = 0.8$	$x = 1.2$	$x = 1.6$	$x = 2$
$t = 0.32$	1.22244	2.26750	3.04680	3.52167	3.68072
$t = 0.64$	1.07921	2.03012	2.76025	3.21469	3.36848
$t = 0.96$	0.96695	1.82940	2.50214	2.92680	3.07165

Take $h = 0.4$, $k = 0.16$, so $r = \frac{k}{h^2} = 0.5$. (5)

The solutions of equation (3) is shown in Table-1 and corresponding analytic values in Table -2. Crank Scheme, exact solution, Percentage error are shown in Table-3.The solutions of equation (4) are shown in Table-4. Modified Scheme, exact solution, Percentage error is shown in Table-5. Comparison of percentage Error by C-N Method and Modified Implicit method is shown in Table 6.

The 2-d and 3-d graph of Modified Implicit Scheme is shown in Fig -1 (a) and Fig-2 (b).From $i = 6,7,8,9,10$ values are symmetric

Test Example-2: Solve the partial differential heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$ with initial condition $u(x,0) = 2x$, $0 \le x \le \frac{1}{2}$, $u(x,0) = 2(1-x)$, $\frac{1}{2} \le x \le 1$ and Boundary condition $u(0,t) = 0$, $u(1,t) = 0$ for $t > 0$.

Solution:

Analytic solution of this partial differential equation is

$$u(x,t) = \frac{8}{\pi^2}\sum \frac{1}{n^2} sin\left(\frac{n\pi}{2}\right) sin(n\pi x)e^{-n^2\pi^2 t}$$

Take $h = 0.1$, $k = 0.005$, so $r = \frac{k}{h^2} = 0.5$. (6)

The solutions of equation (3) is shown in Table-7 and corresponding analytic values in Table -8. Crank Scheme, exact solution, Percentage error are shown in Table-9.The solutions of equation (4) are shown in Table-10. Modified Scheme, exact solution, Percentage error is shown in Table-11. Comparison of percentage Error by C-N Method and Modified Implicit method is shown in Table 12.

The 2-d and 3-d graph of Modified Implicit Scheme is shown in Fig -3 (a) and Fig-4 (b). From $i = 6,7,8,9,10$ values are symmetric

Table-1

$t\backslash x$	$x = 0.4$	$x = 0.8$	$x = 1.2$	$x = 1.6$	$x = 2$
$t = 0.32$	1.213726	2.262744	3.045053	3.521105	3.680432
$t = 0.64$	1.072315	2.022097	2.754186	3.211004	3.365717
$t = 0.96$	0.961034	1.820627	2.493443	2.919484	3.065038

Table-2

$t\backslash x$	$x = 0.4$	$x = 0.8$	$x = 1.2$	$x = 1.6$	$x = 2$
$t = 0.32$	1.209555	2.258173	3.042568	3.520246	3.680031
$t = 0.64$	1.068339	2.016439	2.74928	3.207644	3.36307
$t = 0.96$	0.957384	1.814759	2.487137	2.913706	3.059606

Table-3

t	C-N Method	Exact solution	% Error
$t = 0.32$	1.213726	1.209555	1.0
$t = 0.64$	1.072315	1.068339	1.0
$t = 0.96$	0.961034	0.957384	1.0

IFERP
connecting engineers... developing research

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)

Vol 8, Issue 9, September 2021

Table-4

$t\backslash x$	$x = 0.4$	$x = 0.8$	$x = 1.2$	$x = 1.6$	$x = 2$
$t = 0.32$	1.22244	2.26750	3.04680	3.52167	3.68072
$t = 0.64$	1.07921	2.03012	2.76025	3.21469	3.36848
$t = 0.96$	0.96695	1.82940	2.50214	2.92680	3.07165

Table-5

t	Modified Method	Exact Solution	%error
$t = 0.32$	1.222442	1.209555	0.3
$t = 0.64$	1.079216	1.068339	0.3
$t = 0.96$	0.966955	0.957384	0.3

Table-6

t	% error of Modified method	% error of C-N method
$t = 0.32$	0.3	1.0
$t = 0.64$	0.3	1.0
$t = 0.96$	0.3	1.0

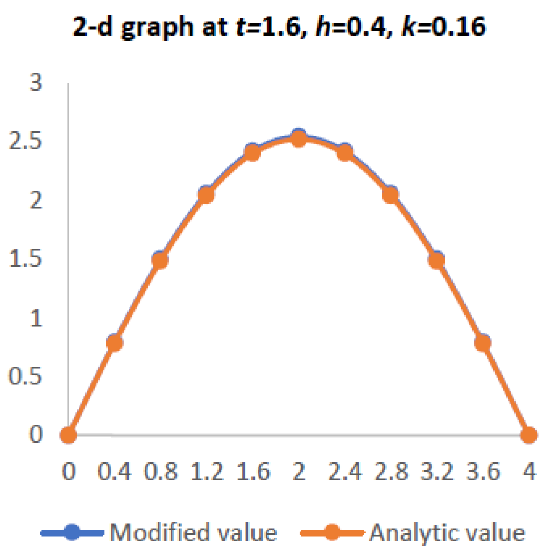

Fig-1 (a)

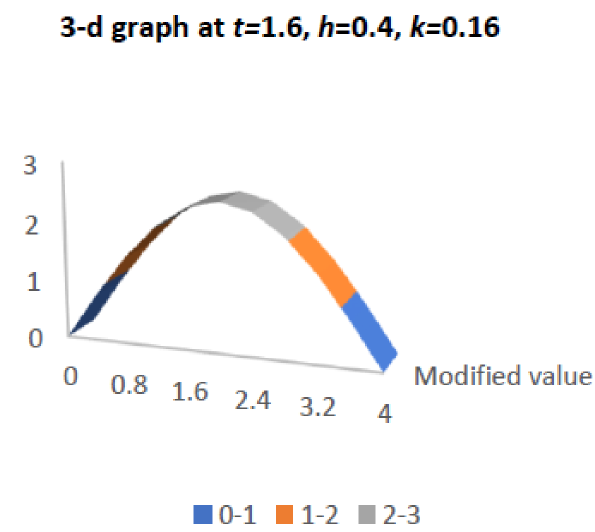

Fig-2 (b)

Table-7

$t\backslash x$	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
$t = 0.025$	0.18818	0.36468	0.51377	0.61556	0.65199
$t = 0.03$	0.18194	0.35061	0.49041	0.58408	0.61721
$t = 0.04$	0.16795	0.32146	0.44587	0.52741	0.55586

Table-8

$t\backslash x$	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
$t = 0.025$	0.18787	0.362967	0.509289	0.60808	0.643177
$t = 0.03$	0.181235	0.348377	0.485749	0.577018	0.609128
$t = 0.04$	0.166695	0.318585	0.441072	0.520966	0.548763

Table-9

t	C-N Method	Exact solution	% Error
$t = 0.025$	0.36468	0.362967	0.4
$t = 0.03$	0.35061	0.348377	0.6
$t = 0.04$	0.32146	0.318585	0.9

Table-10

$t\backslash x$	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
$t = 0.025$	0.18707	0.36329	0.51376	0.61436	0.65211
$t = 0.03$	0.18106	0.34977	0.49115	0.5813	0.61511
$t = 0.04$	0.16756	0.32012	0.44724	0.52581	0.55181

ISSN (Online) 2394-2320

International Journal of Engineering Research in Computer Science and Engineering (IJERCSE)

Vol 8, Issue 9, September 2021

Table-11

t	Modified Method	Exact solution	% Error
$t = 0.32$	0.36329	0.362967	0.0
$t = 0.64$	0.34977	0.348377	0.4
$t = 0.96$	0.32127	0.318585	0.8

Table-12

t	% error of Modified Method	% error of C-N Method
$t = 0.32$	0.0	0.4
$t = 0.64$	0.4	0.6
$t = 0.96$	0.8	0.9

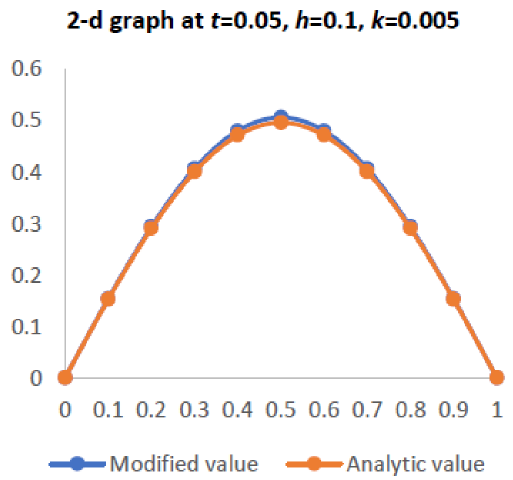

Fig-3 (a)

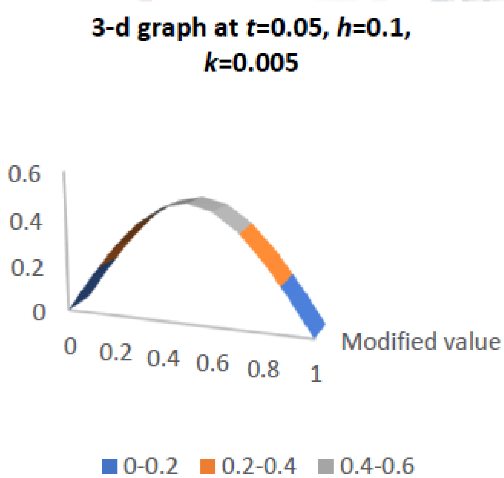

Fig -4 (b)

Test Example-3: Solve the partial differential heat equation $\partial u/\partial t=\partial^2 u/\partial x^2$,$0\le x\le 1$ with initial condition $u(x,0)=sin\pi x$ and boundary conditions $u(0,t)=0$, $u(1,t)=0$ for $t>0$.

Test Example-4: Solve the partial differential heat equation $\partial u/\partial t=\partial^2 u/\partial x^2$,$0\le x\le 1$ with initial condition $u(x,0)=x(1-x)$ and Boundary condition $u(0,t)=0,u(1,t)=0$ for $t>0$.

VI. CONCLUSION

Example 1,2,3,4 are solved for computing approximate solution of one dimensional heat equation with initial and boundary conditions by crank Nicolson Method and our Modified Implicit Method. We proposed a modified Implicit Method for solving one dimensional heat Equation with initial and boundary conditions and this method is valid for 0<r≤1.0. Test Examples solved using modified implicit method Give good approximations to exact solutions of Parabolic partial differential equations. We also Observed that percentage errors computed in test Examples which are solved by our method are less Than those percentage errors in respective test Examples solved by crank Nicolson Implicit Method.

REFERENCES

[1] Ames, W.F., Numerical Methods for partial Differential Equations, Thomas Nelson, London, 1969.
[2] C. E. Abhulimen and B. J. Omowo "Modified Crank Nicolson Equation for Solving one Dimensional Parabolic Equation" IOSR .Volume 15, 2019, pp. 60-66.
[3] Crank J. and Nicolson P "A practical Method For Numerical Evaluation of Solution of Partial Differential Equation of heat Conduction type" proc. Camb. Phil. Soc, Vol 43, pp. 50-67
[4] D.J. Duffy, Finite difference Method in Financial Engineering, A partial differential Equation approach, Wiley 2006.
[5] Froberg, C. E., Introduction to Numerical Analysis, Addison –Wesley, Reading, Massachusetts, 1965.
[6] Hamzat A. Isede, Department of Mathematical Sciences, Redeemer's University, Nigeria. "Several Examples of Crank Nicolson Method for parabolic Partial Differential Equation" Academic Journal of Scientific Research, May 2013
[7] Hildebrand, F.B., Introduction to Numerical Analysis, Tata McGraw-Hill, New Delhi
[8] J.D Hoffman Numerical Methods for Engineers and Scientist, McGraw Hills, New York.

[9] J.D. Smith Numerical solution of P.D.E, London oxford University press 1965.
[10] Kreyszing Advance Engineering Mathematics, USA, John Wiley and son's page 861- 865.
[11] K.W. Morten and D. F. Mayer's Numerical solution of Partial differential equation: An Introduction, Cambridge England 1994.
[12] Mitchel A.R and Griffiths D.F a Finite difference Method in partial Differential Equation, John Wiley and Sons (1980).
[13] Numerical Methods for Scientist and Engineers by Shankar Rao. Chapter 9, pp. 188- 217.
[14] Numerical analysis by Bupendra Singh. Chapter 13 pp. 553-576
[15] Peaceman, D.W. and Rachford, H.H., 'The Numerical Solution of parabolic and elliptic Differential equation' J. Soc. Indust. Appl. Maths., Vol. 3, pp.28-41, 1955.
[16] Samuel Chilabon Zelibe, Federal University of Petroleum Research and Sunday Fadugba, Ekiti State University September 2013, "Crank Nicolson Method For Solving Parabolic Partial Differential Equation".
[17] Spiegel MR (1971) Advance Mathematics for Engineering and Science- Schaum's Outline series, New York McGraw-Hill Book pp-281.
[18] S .P. Frankel and E. C. Du Fort, Conditions in Numerical Treatment of Parabolic Differential Equation, Mathematical Tables and other Aids to Computation, 135-152.
[19] Textbook Notes for parabolic differential equation, Chapter 4.