

# Simulation of Special Mathematical Functions

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**Abstract** - Fractional order system is extensively studied over past few decades. Many simulation and implementation studies have proved that the fractional-order (FO) systems are better than the conventional integer-order systems in terms closed-loop performance. Special mathematical function is unavoidable in most of the solution of fractional order calculus. Computation of these functions requires a large memory and computational resource. Real time calculation is required to speed up performance of a fractional order System. In this work an attempt to simulate the computation of Gamma function, Error function and Complementary error function using ModelSim is made. This work also presents computational error.

**Index Terms**— Fractional Calculus, Special Functions, ModelSim Simulation, FPGA, Hardware Resource Utilization.

## INTRODUCTION

On commencement of learning calculus, one starts with defining the differentiation and integration of a function. Fractional calculus is a branch of mathematics which takes power of derivative or integration as the real numbers or real fractional number or complex number. Leibniz raised a question for non-integer derivative in 1695 led to concept of fractional calculus[1]. Abel used fractional calculus for the solution of tautochronous problem[2]. The problem was to determine the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass  $m$  can fall in a time that is independent of the starting position. This was the first application of the fractional calculus. Fractional calculus is three century old topic but has endless applications in field of engineering like Fluid mechanics, Bio-medical, Quantum mechanics, System modelling, Image processing and Computer vision, Artificial intelligence, efficient and accurate controlling of plant etc. Fractional calculus is used in modeling of speech signal. A paper by K. Assaleh and W.M. Ahmad showed speech signal can be accurately modeled using few integrals of fractional order as basic function[3]. It is also used for modelling the cardiac tissue electrode interface[4]. In image processing integer order differential operator are used for edge detection[5], shows non-integer (fractional) differentiation can improve edge detection of thin detection or case of parabolic luminance. Fractional-order modelling of neutron transport in a nuclear reactor gave a reliable and much realistic model compared to conventional integer-order diffusion model [6].

A deep knowledge of Gamma function, Beta function, Dawson function, Error function and Complementary error function, Gauss Hypergeometric function, Mittag Leffler Function (MLF) is required to understand

definition and use of fractional calculus. Gamma function is most common function. It finds its application in field of Astrophysics, Fluid mechanics, Quantum theory. It is used as a statistical function such as to estimate time between occurrence of earthquake[7].

Mathematical applications of Gamma is to evaluate integration of  $f(t)e^{-g(t)}$ , for example Gamma function is used to evaluate integration of  $t^b \cdot e^{-at}$  as shown in the equation 1.

$$\int_0^{\infty} t^b e^{-at} dt = \frac{\Gamma(b+1)}{a^{b+1}} \quad (1)$$

Gamma function is also used for calculating volume of  $n$  dimension hemisphere. Analytical function can be written as infinite product and these can be represented as finite product or quotient of Gamma function. For example  $\sin(\pi z)$  is given as equation 2.

$$\sin(\pi z) = \frac{\pi z}{\Gamma(z+1)\Gamma(1-z)} \quad (2)$$

The probability of error or Bit Error Rate (BER) of Polar NRZ, BPSK, QPSK, is given as shown in equation 3.

$$BER = 1 - \frac{1}{2} \operatorname{erf} \left( \sqrt{\frac{eb}{no}} \right) \quad (3)$$

In equation 3,  $\frac{eb}{no}$  is ratio of energy per bit to noise and  $\operatorname{erf}(x)$  is Error function defined by equation 7. Error function also appears in solution of heat equation when boundary condition is heavy side step function. Gamma function, Error function and Complementary error

function are used in analytical solution of FO calculus . In this paper, results of ModelSim simulation of Gamma function, Error function and Complementary error function is presented.

## II. SPECIAL MATHEMATICAL FUNCTION

Equation 4 gives the definition of Gamma function for an integer[8].

$$\Gamma(n) = 1 * 2 * 3 * 4 \dots * (n-1)$$

i.e  $\Gamma(n) = (n-1)!$  (4)

In 1738, Euler published the generalized factorial given in equation 5 [9].

$$z! = \int_0^1 (-\ln(s))^z ds \quad (5)$$

The notation  $\Gamma(x)$  was used by Adrien-Marie Legendre in 1811 and extended the integral definition by substituting  $t = -\log$ .

$$\Gamma(z) = \int_0^{\infty} x^{(z-1)} e^{-x} dx \quad (Re(z) > 0.)$$

This definition is commonly used today. Error function is twice the integral of normalised Gaussian function between 0 to  $x/\sigma\sqrt{2}$

$$erf(x) = \int_0^{\frac{x}{\sigma\sqrt{2}}} G(x) dx \quad (6)$$

$$\text{where, } G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

here,  $\sigma^2$  is variance.

hence error function is defined as in equation 7.

$$erf(x) = -\frac{2}{\pi^2} \int_0^x e^{-t^2} dt \quad (7)$$

Complimentary error function is defined as in equation 8

$$erf(x) = -\frac{2}{\pi^2} \int_x^{\infty} e^{-t^2} dt \quad (8)$$

Currently available software for special function exhibits gaps in comparison to the needs of modern high performance real-time scientific implementation. According to papers, fractional-order systems have been studied extensively over the past two decades, and much progress has been made in the theory and analysis of these systems. In general, the implementation of fractional-order systems requires that they be approximated as high order rational systems. As a result, they are difficult to translate into hardware.

## III. DESIGN FLOW

In this paper, for computation of a function is built using Mathwork's Simulink. A Simulink system is built using DSP Builder Blockset which is provided by Altera (Intel) for algorithm designing.

Figure 1 is the system for Error function. Approximation used to build this system is given in section III-B. The "Control" subsystem generates logical control for the system. "coeff" subsystem selects coefficient depending on value of x, while "power" subsystem computes values of the power of x required for Chebyshev polynomial. "calc1" calculates Rational polynomial given by equation 15. "postcalc" multiplies

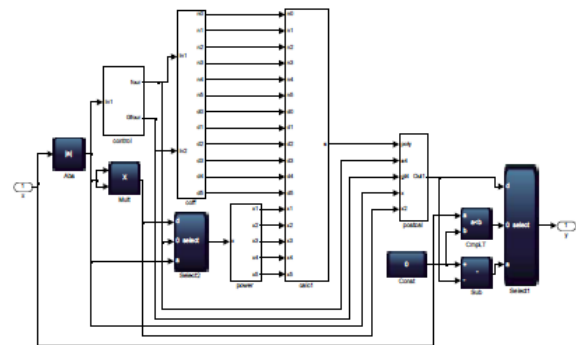
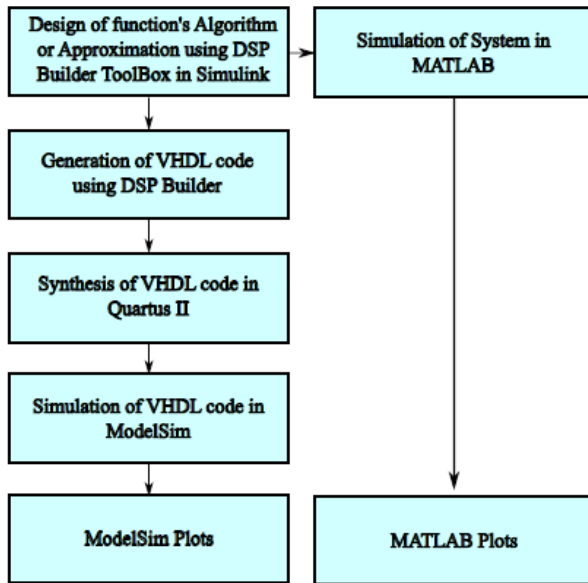


Fig. 1. Simulink System of Error function



**Fig. 2. Design Step for computation**

$Rim(x)$  by  $x, e^{-x^2}, \frac{e^{-x^2}}{x} \left\{ \frac{1}{\sqrt{\pi}} + \frac{1}{x^2} \right\}$  for range  $|x| \leq 0.5, 0.5 < |x| \leq 4, |x| > 4$  respectively.

After this step Mathwork's Simulink automatically generates VHDL code at the run time at the specified file path. This VHDL code can be compiled in Altera's Quartus II. After the successful compilation of the Code it can be simulated in ModelSim environment for computation error. Flow used in this paper to implement is as shown in the figure 2.

#### A. Gamma function

W. J. Cody in 1976 published a paper named "An Overview of Software Development for Special Functions" gave an algorithm to calculate Gamma Function [10]. This algorithm was optimized for using in single or double precision IBM machine. It recommended use of Clenshaw's approximation for  $1/\Gamma(x)$  in range  $1 \leq x \leq 2$  [11] and for range  $0 \leq x < 1$  and  $2 < x \leq 12$  the property  $\Gamma(x+1) = x \Gamma(x)$  to be used and recommended Hart, et al's approximation in range  $x > 12$  [12]. For negative values use of reflection formula  $\Gamma(x) = \frac{\pi}{\sin(\pi x) \Gamma(1-x)}$  was recommended. Evaluation  $y = \Gamma(x)$  is as follows [10]:

If  $x$  is in range 1 to 2 then  
(Calculate  $\Gamma(x)$  by Clenshaw's approximation and is given as:)

$$\Gamma(x) = \frac{P_8 x^8 + P_7 x^7 + \dots + p_1 x}{z^8 + q_7 x^7 + \dots + q_2 x + q_1} + 1 \quad (9)$$

If  $0 < x < 1$  then

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \quad (10)$$

Here, range of  $x$  is from 0 to 1 then  $x+1$  is in range 1 to 2 this can solved by using 9

If  $2 < x < 12$  then

$$\Gamma(x+1) = x \Gamma(x) \quad (11)$$

This can be iterated till  $x$  will be in range 1 to 2 and again  $\Gamma(x)$  is calculated by 9

If  $x \leq 12$  then

This is approximated using Hart, et al's approximation which is stated as:

$$P = c_7 + c_6 x^2 + c_5 x^4 + c_4 x^6 \dots + c_1 x^{12} \quad (12)$$

$$G = \frac{P}{x} - x + 0.9189285332046727417803297 +$$

$$(x + 0.5) \ln(x)$$

$$\Gamma(x) = \exp(G)$$

If  $x < 0$  then

Gamma function is derived by rearranging terms of reflection formula and is given as:

$$\Gamma(-x) = \frac{\pi}{\sin(\pi(x-1)) \Gamma(x-1)} \quad (13)$$

#### B. Error function

W. J. Cody in 1969 gave an approximation for Error function and Complementary error function [13]. The algorithm used rational polynomial to compute Error function for different ranges. The equation for Error function is as shown in 14.

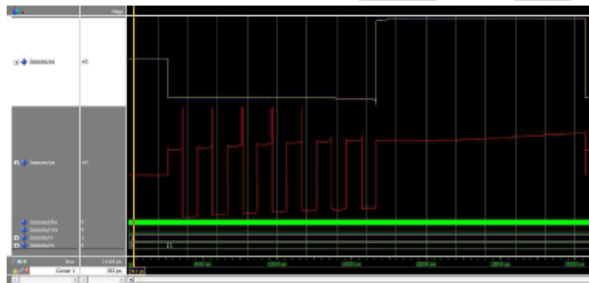
$$erf(x) = \begin{cases} xR_{lm}(x^2) & |x| \leq 0.5 \\ 1 - e^{-x^2} R_{lm}(x) & 0.5 \leq |x| \leq 4 \\ 1 - \frac{e^{-x^2}}{x} \left\{ \frac{1}{\sqrt{\pi}} + \frac{1}{x^2} R_{lm}\left(\frac{1}{x^2}\right) \right\} & x \geq 4 \end{cases} \quad (14)$$

Here,  $R_{lm}(x)$  is a rational function of degree  $l$  in the numerator and  $m$  in denominator, given as:

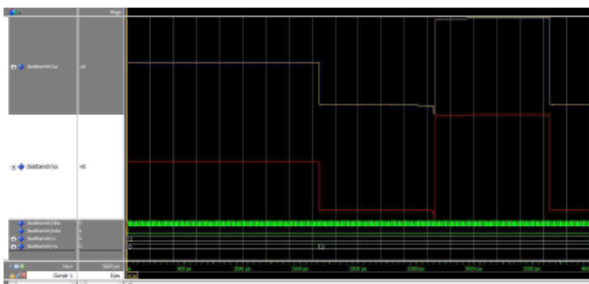
$$R_{lm}(x) = \frac{\sum_{j=0}^l p_j x^j}{\sum_{j=0}^l q_j x^j} \quad (15)$$

For the complementary error function relation between Error and Complementary error function given by equation 16 can be used.

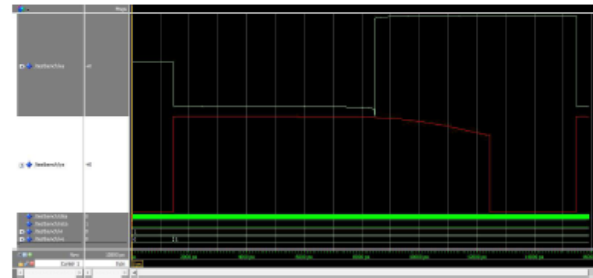
$$erfc(x) = 1 - erf(x) \quad (16)$$



(a) Gamma function



(b) Error function



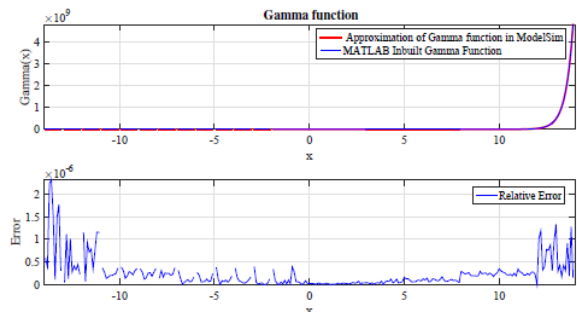
(c) Complementary Error function  
Fig. 3. ModelSim waveform

#### IV. RESULT

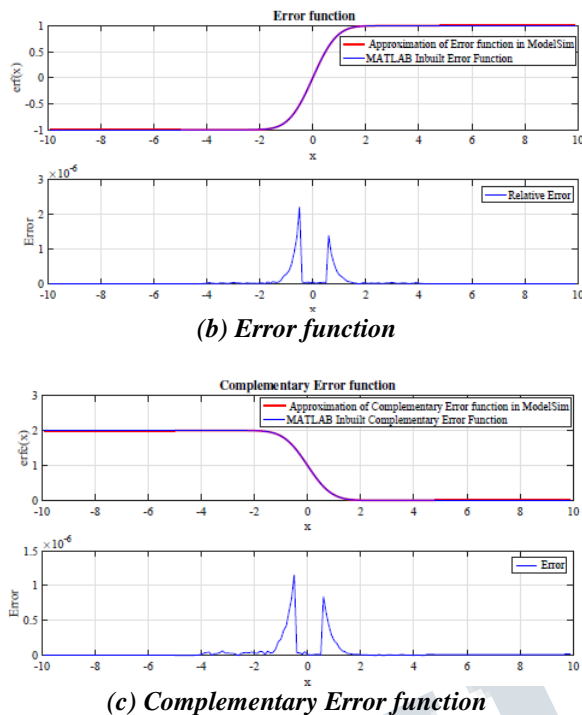
The ModelSim plot of Gamma function Error function and Complementary error function is in figure 3 (a), (b) and (c) respectively. These functions are imported to MATLAB and a comparison is made in terms of relative error with MATLAB's inbuilt Functions. Figure 4 gives the MATLAB's plot. Relative error is calculated using equation 17. From table I, it is evident that the simulation of these function gives precise result. The computation Error is of order  $10^{-6}$  which is close enough with the epsilon of the machine which is operated under single precision floating point format.

**TABLE I**  
**RELATIVE ERROR BETWEEN MODELSIM AND**  
**MATLAB'S INBUILT FUNCTION**

Functions	Minimum relative error	Maximum relative error
$\Gamma(x)$	0	2.321847e-06
$erf(x)$	0	2.217498e-06
$erfc(x)$	0	1.154208e-06



(a) Gamma function



**Fig. 4. MATLAB and ModelSim data plot of function**

$$\text{Relative error} = \frac{\text{modelSim Value} - \text{Inbuilt value}}{\text{Inbuilt Value}} \quad (17)$$

## V. CONCLUSION

Special mathematical functions important in fractional calculus is simulated in ModelSim environment in this work. A systematic approach to simulate four mathematical special functions is presented. The results clearly show the successful simulation of these functions on ModelSim. This work can further be extended to implement the functions on an FPGA. This work can be used to design a fractional order controller on an FPGA.

## REFERENCES

- [1] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu-Batlle, "Fractional-order Systems and Controls".
- [2] A. N.H, "Solution de quelques a laide d'integrales defines," Ocuvers Completes, vol. 1, 1881.

[3] K. Assaleh and W. Ahmad, "Modeling of speech signals using fractional calculus," 9th International Symposium on Signal Processing and Its Applications, vol. 3, pp. 1–4, 2007.

[4] R. L. Magin, "Modeling the cardiac tissue electrode interface using fractional calculus," Journal of Vibration and Control, vol. 14, no. 9–10, pp. 1431–1442, 2007.

[5] B. Mathieu, P. Melchior, A. Oustaloup, and C. Ceyral, "Fractional differentiation for edge detection," Fractional Signal Processing and Applications, vol. 14, no. 23, pp. 9747 – 9767, 2007.

[6] V. A. Vyawahare and P. Nataraj, "Fractional-order modeling of neutron transport in a nuclear reactor," Applied Mathematical Modelling, vol. 37, no. 9–10, pp. 1431–1442, 2013.

[7] J. A. Rice, "Mathematical statistics and data analysis," Applied Mathematical Modelling, vol. 2, pp. 52–53, 1995.

[8] E. A. translated by Michael Butler, The Gamma Function. New York: Holt, Rinehart and Winston, 1964.

[9] P. J. Davis, "Leonhard Euler's Integral: An Historical Profile of the Gamma Function," The American Mathematical Monthly, vol. 66, pp. 849–869, 1959.

[10] W. J. Cody, "An overview of software development for special functions," Proceedings of the Dundee Conference on Numerical Analysis, pp. 38–48, 1975.

[11] C. C.W., M. G.F., and W. M, "Numerische Mathematik". Springer, 1962, vol. 4.

[12] J. F. Hart, E. W. Cheney, C. L. Lawson, H. J. Maehly, C. K. Mesztenyi, J. R. Rice, H. C. Thacher, and C. Witzgall, "Computer Approximations". John Wiley & Sons Inc., 1964.

[13] W. J. Cody, "Rational chebyshev approximations for the error function," Mathematics of Computation, vol. 23, no. 107, pp. 631– 637, 1969. [Online]. Available: <http://www.ams.org/mcom/1969-23-107/S0025-5718-1969-0247736-4/>