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Transient Thermal Stresses of an Elliptical Disc with Internal Heat Generation

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Abstract: **-- In this paper, we deal with transient thermal stress problem of an elliptical disc subjected to internal heat source with mixed-type boundary conditions. The solution of conductivity equation and the corresponding initial and boundary conditions are obtained by employing a new integral transform technique. The governing equation for small deflection is found and utilized to find intensities of thermal bending moments and twisting moments, etc. involving the Mathieu and modified functions and their derivatives. The analytical solution for the thermal stress components are obtained in terms of resultant forces and resultant moments, and same are illustrated graphically.**

Keywords: -- Elliptical annulus plate, temperature distribution, thermal stresses, Mathieu function, and thermal moment.

I. INTRODUCTION

Thermo elastic problems with mixed-type boundary conditions occur frequently in engineering applications. Mahalanabis [3] obtained stress and displacements in the case of transversely isotropic material for a half-space and a thick plate when the temperature is prescribed on a circular region of the free surface and the remainder radiates in accordance with Newton's Law.Singh [4] has considered two mixed boundary-value problems, (i) steady-state plane-strain thermoelastic problem, and (ii) electrostatic problem having potential,and reduced both to the solution of triple trigonometric integral equations.Huang [5] constructed Green's function of anisotropic heat conduction on a thin plate under steady state two-dimensional domain and applied it to an elliptical disc. Though,bending problems forfixed and clamped elliptical plates under the action of various external forces got wide consideration for practical applications in aircraft structures during the past years. Solution for elliptical thin plates bent by a concentrated moment placed at the center of the plate is sought for built-in edges and for fixed and clamped edges have been investigated by Cheng [2] with the aid of tensor calculus in which the expressions for moments, shearing forces and the biharmonic equation for deflection were transformed into elliptic coordinates. Pravin [6] investigated the thermo-elastic problems on an elliptical plate in which interior heat sources are generated within the solid, with compounded effect due to sectional heating and boundary conditions of the Dirichlet type based on the theory of integral transformations. But till date nobody has studied any thermo-elastic problem for elliptical plates with boundary conditions as mixed type, in which plates are considered to be fixed and clamped

II. FORMULATION OF THE PROBLEM

It Is Assumed That A Thin Annulus Elliptical Plate Occupying the Space $D:\{(\xi,\eta,z)\in R^3 : a<\xi<b,\ 0<\eta<2\pi,\ 0<z<\ell\}$ under unsteady-state temperature field due to internal heat source within it.the geometry of the plate indicates that an elliptic coordinate system (ξ,η,z) is the most appropriate choices of reference frame, which are related to the rectangular coordinate system (x,y,z) by the relation $x=c\cosh\xi\cos\eta,\ y=c\sinh\xi\sin\eta,\ z=z$. The Curves $\eta=$Constant Represent A Family Of Confocal Hyperbolas While The Curves $\xi=$Constant Represent A Family Of Confucian Ellipses(Refer To Figure 1).Both Sets Of Curves Intersect Each Other Orthogonally At Every point in space. the geometry parameters are given as $\xi\in[a,b],\eta\in[0,2\pi)$ and $z\in(0,\ell)$.

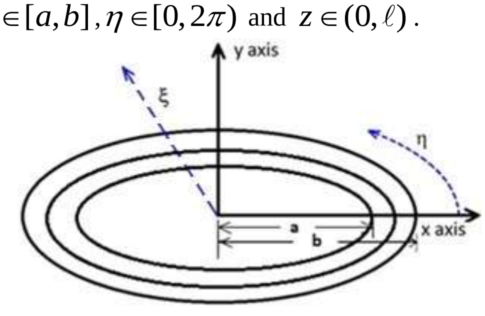

Figure 1.Platephysical Configuration

The heat conduction equation and boundary conditions are given as

$$h^2\left\{\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2}\right\}+\frac{\partial^2\theta}{\partial z^2}+\frac{Q(\xi,\eta,t)}{\lambda}=\frac{1}{\kappa}\frac{\partial\theta}{\partial t} \quad (1)$$

Where $\theta(\xi,\eta,z,t)$ is temperature of the plate at point (ξ,η,z) at t time, λ is the coefficient of thermal conductivity, $Q(\xi,\eta,t)$ represents an energy generation term, $\kappa=\lambda/\rho C$ represents thermal diffusivity in which λ being the thermal conductivity of the material, ρ is the density, C is the calorific capacity, assumed to be constant and $h^{-2}=(c^2/2)(\cosh 2\xi-\cos 2\eta)$.

the heat generation term is assumed in the form

$$Q(\xi,\eta,z,t)=Q_0\delta(\xi-a_0)\delta(\eta-2\pi)\delta(z-\ell_0) \quad (2)$$

in which Q_0 characterizes the stream of heat, $\delta(\;)$ is the dirac delta function in which

$\xi\neq a_0, a_0\in[0,a]$ and $z\neq\ell_0$, $z\in[0,\ell]$.

the temperature distribution in the elliptical plate is obtained as a solution of the equation (1) with the following initial and boundary conditions

$$\theta(\xi,\eta,z,0)=0 \quad (3)$$

$$\frac{\partial\theta}{\partial\xi}(a,\eta,z,t)=f(z,t) \quad (4)$$

$$\frac{\partial\theta}{\partial\xi}(b,\eta,\ell,t)+\theta(b,\eta,\ell,t)=g(z,t) \quad (5)$$

$$\theta(\xi,\eta,0,t)=0 \quad (6)$$

$$\theta(\xi,\eta,\ell,t)=0 \quad (7)$$

The Most General Form Of The Equation Of Equilibrium For A Plate Element Is Expressed In terms of the partial derivatives of the deflection is found to satisfy the differential equations as

$$D\nabla^4\omega+\frac{\nabla^2 M_\theta}{1-\upsilon}=0 \quad (8)$$

where D is the stiffness coefficient of the plate and denoted as

$$D=\frac{E\ell^3}{12(1-\upsilon^2)} \quad (9)$$

and M_θ is bending of the plate due to change in the temperature and expressed as

$$M_\theta=\alpha E\int_0^\ell z\,\theta\,dz \quad (10)$$

in which ∇^2 denotes the two-dimensional laplacian operator in (ξ,η), υ denotes poisson's ratio, α and E denoting coefficient of linear thermal expansion and young's modulus of the material of the plate respectively. in order to complete the formulation of the problem, it is necessary to introduce suitable boundary conditions. the plate edgeis assumed to be fixed and clamped, that is

$$\omega(\xi,\eta,0)=0,\ \omega(a,\eta,t)=\frac{\partial\omega(b,\eta,t)}{\partial\xi}=0 \quad (11)$$

the equations (1) to (16) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION FOR THE PROBLEM

In order to solve fundamental differential equation (1), firstly we introduce the new integral transformation of order *n* and *m* over the variable ξ and η as

$$\bar f(q_{2n,m})=\int_a^b\int_0^{2\pi} f(\xi,\eta)(\cosh 2\xi-\cos 2\eta)\times M_{2n,m}(k,\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})\,d\xi\,d\eta$$

(12) ***The Inversion Theorem Of (12) Is***

$$f(\xi,\eta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\bar f(q_{n,m})M_{2n,m}(k,\xi,\eta,q_{n,m})ce_{2n}(\eta,q_{2n,m})/C_{2n,m} \quad \mathbf{(13)}$$

where, the kernel is given as

$$M_{2n,m}(k,\xi,\eta,q_{n,m})=\{[fey_{2n}(a,q_{2n,m})+B_{2n}(k,b,q_{2n,m})]Ce_{2n}(\xi,q_{2n,m})-[Ce_{2n}(a,q_{2n,m})+A_{2n}(k,b,q_{2n,m})]fey_{2n}(\xi,q_{2n,m})\}$$

in which $q_{2n,m}$ is a root of transcendental equation

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$$Ce_{2n}(a,q_{2n,m})B_{2n}(k,b,q_{2n,m}) - fey_{2n}(a,q_{2n,m})A_{2n}(k,b,q_{2n,m}) = 0,$$

$$A_{2n}(k,b,q_{2n,m}) = Ce_{2n}(b,q_{2n,m}) + k\,Ce'_{2n}(b,q_{2n,m}),$$

$$B_{2n}(k,b,q_{2n,m}) = fey_{2n}(b,q_{2n,m}) + k\,fey'_{2n}(b,q_{2n,m}),$$

$$C_{2n,m} = \pi \int_a^b (\cosh 2\xi - \Theta_{2n,m}) M^2_{2n,m}(\xi, q_{2n,m})\, d\xi$$

and operational property as

$$\int_a^b\int_0^{2\pi}\left(\frac{\partial^2 f}{\partial\xi^2}+\frac{\partial^2 f}{\partial\eta^2}\right)M_{2n,m}(k,\xi,\eta,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})\,d\xi\,d\eta = -2q_{2n,m}\bar{f} \quad \textbf{(14)}$$
$$+\left(\ell\frac{\partial f}{\partial\xi}+f\right)\Bigg|_{\xi=b} M_{2n,m}(k,b,\eta,\ell,q_{2n,m}) - \left(\frac{\partial f}{\partial\xi}\right)\Bigg|_{\xi=a} M'_{2n,m}(k,a,\eta,\ell,q_{2n,m})$$

the kernel of above transform is indicated in elliptical function and it removes the variable ξ and $\eta$from the differential equation defined in (1) for the mixed-type boundary conditions given in equations (5) and (6). on applying new integral transform in equation (12) to the differential equation (1), and taking the property (14) into account under the conditions (5) and (6), the differential equation for $\theta(\xi,\eta,z,t)$ is reduced to

$$\frac{\partial^2\bar{\theta}}{\partial z^2} - \frac{4q_{2n,m}}{c^2}\bar{\theta} + \Omega(z,t) + \frac{\bar{Q}}{\lambda} = \frac{1}{\kappa}\frac{\partial\bar{\theta}}{\partial t} \quad \textbf{(15)}$$

in which

$\Omega(z,t) = 2\alpha\pi A_0^{(2n)}\{M_{2n,m}(b,q_{2n,m})\,g(z,t) - M'_{2n,m}(a,q_{2n,m})\,f(z,t)\}$ then,

applying fourierfinite sine transform to the equation (15) and taking into account the boundary conditions (6) and (7), the differential equation for $\bar{\theta}(q_{2n,m},z,t)$ is transformed into

$$\frac{\partial\bar{\theta}^*}{\partial t} + \alpha^2_{2n,m}\bar{\theta}^* = \Omega^*(\beta_p,t) + \bar{Q}^* \quad \textbf{(16)}$$

where

$$\alpha_{2n,m} = \kappa(4q_{2n,m}/c^2 + \beta_p{}^2),$$

and β_p is a positive root of transcendental equation

$\sin(\beta_p\ell) = 0$, $\beta_p = m\pi/\ell$.

thus, temperature solution in the transformed domain reduces to

$$\bar{\theta}^*(q_{2n,m},\beta_p,t) = \int_0^t \exp[-\alpha_{2n,m}(t-\tau)][\Omega^*(\beta_p,t) + \bar{\theta}^*]\,d\tau \quad (17)$$

Finally, using the inversion of the fourier finite sine transform and the inversion theorem (13), we obtain the solution as

$$\theta(\xi,\eta,z,t) = \frac{2}{\ell}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\sum_{p=0}^{\infty}\sin(\beta_p z)\left[\int_0^t \exp[-\alpha_{2n,m}(t-\tau)][\Omega^*(\beta_p,t)+\bar{\theta}^*]\,d\tau\right]\right\} \quad (18)$$
$$\times M_{2n,m}(\xi,k,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})/C_{2n,m}$$

the above function given in equation (18) represents the temperature at every instance and at all points of elliptical annulus of finite height under the influence of mixed-type boundary conditions.
on substituting equation (18) in (8), one obtains the thermal moment as

$$M_\theta(\xi,\eta,t) = \frac{2\alpha E}{\ell}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\sum_{p=0}^{\infty}\frac{[\sin(\beta_p) - \beta_p\cos(\beta_p\ell)]}{\int_0^t \exp[-\alpha_{2n,m}(t-\tau)]\times[\Omega^*(\beta_p,t)+\bar{\theta}^*]\,d\tau}\right\} \quad (19)$$
$$M_{2n,m}(\xi,k,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m})/C_{2n,m}\,\beta_p^2$$

using equation (19)into equation (8)satisfying boundary conditions (11), we obtain the expression for thermal deflection as

$$\omega(\xi,\eta,t) = \frac{\alpha c^2 E}{2D(1-\upsilon)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\sum_{p=0}^{\infty}[\sin(\beta_p) - \beta_p\cos(\beta_p\ell)]\right.$$
$$\left.\int_0^t \exp[-\alpha_{2n,m}(t-\tau)][\Omega^*(\beta_p,t)+\bar{\theta}^*]\,d\tau\right\}$$
$$Z_{2n,m}M'_{2n,m}(a,k,q_{2n,m})\,ce_{2n}(\eta,q_{2n,m}) \quad \textbf{(20)}$$
$$\times\{a\,Z_{2n,m} - \xi[Z_{2n,m} + M_{2n,m}(\xi,k,q_{2n,m})$$
$$+ M_{2n,m}(a,k,q_{2n,m})]\}/(a\,q_{2n,m}\,C_{2n,m}\,\beta_p^2)$$

where

$$Z_{n,m} = M_{2n,m}(a,k,q_{2n,m}) - M_{2n,m}(b,k,q_{2n,m}) - b\,M'_{2n,m}(a,k,q_{2n,m})$$

IV. NUMERICAL RESULTS, DISCUSSION AND REMARKS

For the sake of simplicity of calculation, we introduce the following dimensionless values

$$\left.\begin{aligned}&\bar{\xi} = \xi/b,\ \bar{a} = a/b,\ \bar{b} = b/b,\ \bar{z} = z/b,\ \bar{\ell} = \ell/b,\ e = c/b,\ \tau = \kappa t/b^2,\\ &\bar{\theta} = \theta/\theta_0,\ \varpi = \omega/E\alpha_t\theta_0 b,\ \bar{\sigma}_{ij} = \sigma_{ij}/E\alpha_t\theta_0\ \ (i,j=\xi,\eta),\end{aligned}\right\}$$

(21)

Substituting the value of equation (21) in equations (18)-(20), we obtained the expressions for the temperature and thermal deflection respectively for our numerical discussion. The numerical computations have been carried out for Aluminum metal with parameter a= 1 cm, b= 2 cm,h= 1.00 cm, Modulus of Elasticity E= 6.9 × 10^6N/cm^2, Shear modulus G= 2.7 × 10^6N/cm^2, Poisson ratioυ = 0.281, Thermal expansion coefficient, α=25.5× 10^6cm/cm-^{0}C, Thermal diffusivityκ= 0.86 cm^2/sec, Thermal conductivityλ = 0.48 calsec^{-1}/cm ^{0}C with $\alpha_{2n,m}$ =0.0986, 0.3947, 0.8882, 1.5791, 2.4674, 3.5530, 4.8361, 6.3165, 7.9943, 9.8696, 11.9422, 14.2122, 16.6796, 19.3444, 22.2066, 25.2661, 28.5231, 31.9775, 35.6292, 39.4784are the positive & real roots of the transcendental equation. In order to examine the influence of heating on the plate, we performed the numerical calculation for all variables and numerical calculations are depicted in the following figures with the help of MATHEMATICA software.

V. CONCLUSION

In this article, we have described the theoretical treatment of temperature distribution and the deflection in the form of ordinary and modified Mathieu functions used to determine thermal stresses by proposed new operational methods. The results mentioned which was obtained while carrying out our research are illustrated as follows,

The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions during large deflection under thermal loading.

REFERENCES

[1] McLachlan, N.W., Theory and Application of Mathieu function, Oxford Univ. Press, 1947.

[2] Cheng, S., Bending of an elliptic plate under a moment at the center, Tech. Sum. Rept. No. 444, Math. Res. Center, Univ, Wisconsin (1963).

[3] Mahalanabis, R.K., On steady state mixed boundary value problems of a transversely isotropic material for a half-space and a thick plate, Bulletin of Cal. Math. Soc., vol. 60, no. 4, pp. 151-162, 1968.

[4] Singh, B. M., Dhaliwal, R.S., Mixed boundary-value problems of steady-state thermoelasticity and electrostatics, Journal of Thermal Stresses, vol. 1, no. 1, pp. 1-11, 1978.

[5] Huang, S.C., Chang, Y.P., Anisotropic heat conduction with mixed boundary conditions, J. of Heat Transfer, vol. 106, no. 3. pp. 646–648, 1984.

[6] Bhad, P. P., Varghese, V., Khalsa, L. H., Heat source problem of thermoelasticity in an elliptic plate with thermal bending moments, *Journal of Thermal Stresses* [Accepted for publication in Aug'2016].