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# Application of Collocation Method Using Nurbs Basis Functions for 1-D Heat Transfer Problems

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***Abstract:*-- The present work involves in the use the NURBS basis functions, with varying degree, as the basis functions in Collocation Method. A comprehensive step-by-step procedure for using NURBS Collocation method is developed and documented for applying this method to heat transfer problems. This method is applied to 1-D conductive Heat Transfer through slab. The results obtained with NURBS Collocation Method are closed to exact solution. The solution obtained by collocation method is found to be accurate and far simpler to solve than many available approximate methods.**

**Keywords— Basis functions, NURBS, Collocation method, Heat transfer.**

## 1. INTRODUCTION

Mathematical models in the form of differential and partial differential equations are used to represent various engineering problems in the fields, such as Structural mechanics, Solid mechanics, Fluid flow, Heat transfer, Vibration analyses, Contact mechanics etc. The solutions to these mathematical models can be Exact, Analytical or Approximate depending on the nature of these equations. Numerical techniques are used to solve the mathematical models in engineering problems. Many of the mathematical models of engineering problems are expressed in terms of Boundary Value Problems, which are partial differential equations with boundary conditions. Exact solution is said to be a closed-form solution if it relates a given problem in terms of functions and mathematical operations (Analytical Methods) from a given generally acceptable set. When the Exact solution is not possible, numerical methods are needed to obtain approximate solutions. Two of the most popular techniques for solving these mathematical models which are in the form of partial differential equations are the Finite Element Method (FEM) and the Finite Difference Method [1]. In numerical Methods, the interpolation (shape) functions provide higher order of continuity and are capable of providing accurate solutions with continuous gradients throughout the domain. In the recent years Spline, B-spline, NURBS functions together with some numerical techniques have been used in getting the numerical solution of the differential equations, those techniques are B-Spline Finite Element Method[2],[3], Bezier Finite Element Method, B-Spline Galerkin Approach [ 4], and B-Spline Collocation Method [5],[6],[7][8],[9].

NURBS basis functions are bases for piecewise polynomials that possess attractive properties like they have compact support, yield numerical schemes with high resolving power, with an order of accuracy *p* that is a mere input parameter. In a sense, a NURBS basis can be viewed as a finite-element basis of arbitrary order *p*. They form partition of unity and hence can exactly represent the state of constant field in the domain of interest [10].

In the Present work, NURBS basis functions are used for field variable to solve the boundary value problem. A non uniform knot vector for a particular weight vector is used to obtain second and third degree NURBS basis functions. For special decartelization collocation method is employed. Solutions of one test problem obtained for different degree of NURBS functions is discussed in section 4.

Considering second order linear differential equations with variable coefficients

$$\frac{d^2U}{dx^2}+k_1P(x)\frac{dU}{dx}+k_2Q(x)U=F(x),\quad a\le x\le b \qquad (1)$$

With the boundary conditions $U(a)=d_1$, $U(b)=d_2$. Where $a$, $b$, $d_1$, $d_2$, $k_1$ and $k_2$ are variables, $P(x)$, $Q(x)$ and $F(x)$ are functions of $x$. Let the approximation solution be

$$U^h(x)=\sum_{i=-2}^{n-1}C_iR_{i,p}(x) \qquad (2)$$

Where $C_i$ are constants to be determined and $R_{i,p}(x)$ are NURBS basis functions. $U^h(x)$ is the approximate global solution to exact solution $U(x)$ of the considered second order differential equation (1).

 

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## 2. NURBS BASIS FUNCTIONS:

Non-Uniform Rational B-Splines (NURBS) was introduced by K. Versprille [11] as significant improvement that can accurately handle both analytic and modelled curves. NURBS are used in most computer graphics applications, significantly in CAE and renewed industry standards such as IGES (Initial Graphics Exchange Specification), STEP (Standard for the Exchange of Product model data).

A Rational B-spline curve is the projection of a non-rational B-spline curve defined in four-dimensional (4D) homogeneous coordinate space back into three-dimensional (3D) physical space. Specifically,

$$p(x) = \sum_{i=1}^{n+1} B^{h}{}_{i} N_{i.p}(x) \tag{3}$$

Where $x_{min} \le x \le x_{max}$, $2 \le p \le n+1$, and $B^{h}{}_{i}$ 's are the 4D homogeneous defining polygon vertices for the non-rational 4D B-spline curve. $N_{i,p}(x)$ is the Non-rational B-spline basis function of order $p$ and degree $p-1$ given by

i) If $p=1$

$$N_{i,p}(x) = \begin{cases} 1 & \text{if} \quad x_i \le x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

ii) If $p>1$

$$N_{i,p}(x) = \frac{(x - x_i) N_{i,p-1}(x)}{x_{i+p-1} - x_i} + \frac{(x_{i+p} - x) N_{i+1,p-1}(x)}{x_{i+p} - x_{i+1}} \tag{5}$$

The values of $x_i$ are elements of knot vector satisfying the relation $x_i \le x_{i+1}$. The parameter x varies from $x_{min}$ to $x_{max}$ along the curve p(x).

Projecting back into three-dimensional space by dividing through by the homogeneous coordinate yields the rational B-spline curve.

$$p(x) = \frac{\sum_{i=1}^{n+1} B_i h_i N_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} = \sum_{i=1}^{n+1} B_i R_{i,p}(x) \tag{6}$$

Where the $B_i$ 's are the 3D defining polygon vertices for the rational B-spline curve and the rational B-spline basis functions given by

$$R_{i,p}(x) = \frac{h_i N_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} \tag{7}$$

Here, $h_i$ 's are the homogeneous coordinates (occasionally called weights) provide additional blending capability. It is clear that when all $h_i = 1$, $R_{i,p}(x) = N_{i,p}(x)$, thus non-rational B-spline basis functions and curves are included as a special case of rational B-spline basis functions and curves.

### *2.1 Derivatives of NURBS basis functions*

The first derivative of NURBS curve is

$$p'(x) = \sum_{i=1}^{n+1} B_i R'_{i.p}(x)$$

Where

$$R'_{i,p}(x) = \frac{h_i N'_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} - \frac{h_i N_{i,p}(x) \sum_{i=1}^{n+1} h_i N'_{i,p}(x)}{\left( \sum_{i=1}^{n+1} h_i N_{i,p}(x) \right)^2} \tag{8}$$

The second derivative of NURBS curve is

$$p''(x) = \sum_{i=1}^{n+1} B_i R''_{i.p}(x)$$

where

$$R''_{i,p}(x) = \frac{h_i N''_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} - \frac{R_{i,p}(x) \sum_{i=1}^{n+1} h_i N''_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} - 2 * \frac{R'_{i,p}(x) \sum_{i=1}^{n+1} h_i N'_{i,p}(x)}{\sum_{i=1}^{n+1} h_i N_{i,p}(x)} \tag{9}$$

## 3. COLLOCATION METHOD:

Collocation method is used widely in Approximation theory particularly solving differential equations. The collocation method together with NURBS (Non-Uniform Rational Basis Spline) approximations represents an economical alternative since it only requires the evaluation of the unknown parameters at the grid points or nodes or collocation points. In normal collocation method we use polynomials whereas in NURBS collocation method we use NURBS basis functions. The selection of nodes or collocation points is arbitrary. The basis functions vanish at the boundary values. The success of this Collocation method is dependent on the choice of basis. Our main aim is to analyse the efficiency of the NURBS based collocation method for such problems with sufficient accuracy.

First derivative of approximation function (2) is

$$\frac{dU^h(x)}{dx} = \sum_{i=-2}^{n-1} C_i R'_{i,p}(x) \qquad (10)$$

Second derivative of approximation function (2) is

$$\frac{d^2U^h}{dx^2} = \sum_{i=-2}^{n-1} C_i R''_{i,p}(x) \qquad (11)$$

Substituting, the approximate solution (2) in (1) we have,

$$\frac{d^2U^h}{dx^2} + k_1 P(x)\frac{dU^h}{dx} + k_2 Q(x)U^h = F(x) \qquad (12)$$

Substituting the Approximation function and its derivatives (2), (10) and (11) in the equation (12), we have

$$\sum_{i=-2}^{n-1} C_i R''_{i,p}(x) + k_1 P(x)\sum_{i=-2}^{n-1} C_i R'_{i,p}(x) + k_2 Q(x)\sum_{i=-2}^{n-1} C_i R_{i,p}(x) = F(x) \qquad (13)$$

Expanding the equation (13)

$$[C_{-2}R''_{-2,p}(x)+C_{-1}R''_{-1,p}(x)+C_1R''_{1,p}(x)+.......+C_{n-1}R''_{n-1,p}(x)]+$$
$$k_1P(x)[C_{-2}R'_{-2,p}(x)+C_{-1}R'_{-1,p}(x)+C_1R'_{1,p}(x)+.......+C_{n-1}R'_{n-1,p}(x)]+$$
$$k_2Q(x)[C_{-2}R_{-2,p}(x)+C_{-1}R_{-1,p}(x)+C_1R_{1,p}(x)+.......+C_{n-1}R_{n-1,p}(x)]=F(x) \qquad (14)$$

Now let the coefficients of $C_{-2}$, $C_{-1}$, $C_1$……, $C_{n-1}$ are assumed as $R_{-2}(x)$, $R_{-1}(x)$, $R_1(x)$….$R_{n-1}(x)$, now we have the equation (14) as

$$[R_{-2}(x)]C_{-2}+[R_{-1}(x)]C_{-1}+[R_1(x)]C_1+.......+[R_{n-1}(x)]C_{n-1}=F(x) \qquad (15)$$

Equation (15) in the matrix form, we have

$$[R_{-2}(x) \quad R_{-1}(x) \quad R_1(x). \quad . \quad . \quad .R_{n-1}(x)]\begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ C_{n-1} \end{bmatrix} = F(x) \qquad (16)$$

Equation (16) is evaluated at $x_i$'s, i=1,2,3,….n-1 gives the system of (n-1)×(n+1) equations in which (n+1) arbitrary constants are involved.

Now the Matrix of equation (16) can be written as

$$\begin{bmatrix} R_{-2}(1) & R_{-1}(1) & R_1(1)..... & R_{n-1}(1) \\ R_{-2}(2) & R_{-1}(2) & R_1(2)..... & R_{n-1}(2) \\ R_{-2}(3) & R_{-1}(3) & R_1(3)..... & R_{n-1}(3) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R_{-2}(n-1) & R_{-1}(n-1) & R_1(n-1)..... & R_{n-1}(n-1) \end{bmatrix}\begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} F(1) \\ F(2) \\ F(3) \\ . \\ . \\ . \\ F(n-1) \end{bmatrix} \qquad (17)$$

Two more equations are needed square matrix which helps to determine the (n+1) arbitrary constants. The remaining two equations are obtained using

$$\sum_{i=-2}^{n-1} C_i R_{i,p}(a) = d_1$$

$$\sum_{i=-2}^{n-1} C_i R_{i,p}(b) = d_2 \qquad (18)$$

Now a square matrix of size (n+1) is obtained from equations (17)and (18)

$$\begin{bmatrix} R_{-2}(a) & R_{-1}(a) & R_1(a). \; . \; . \; . & R_{n-1}(a) \\ R_{-2}(1) & R_{-1}(1) & R_1(1). \; . \; . \; . & R_{n-1}(1) \\ R_{-2}(2) & R_{-1}(2) & R_1(2). \; . \; . \; . & R_{n-1}(2) \\ R_{-2}(3) & R_{-1}(3) & R_1(3). \; . \; . \; . & R_{n-1}(3) \\ . & . & .. \; . \; . \; . & . \\ . & . & .. \; . \; . \; . & . \\ . & . & .. \; . \; . \; . & . \\ R_{-2}(n-1) & R_{-1}(n-1) & R_1(n-1). \; . \; . \; . & R_{n-1}(n-1) \\ R_{-2}(b) & R_{-1}(b) & R_1(b). \; . \; . \; . & R_{n-1}(b) \end{bmatrix}\begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ . \\ . \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 \\ F(1) \\ F(2) \\ F(3) \\ . \\ . \\ . \\ F(n-1) \\ d_2 \end{bmatrix} \qquad (19)$$

It is in the form of $[R][C] = [F]$

The matrix ***[R]*** is diagonally dominated square matrix of size (n+1) because every second degree basis

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function has values other than zeros only in three intervals and zeros in the remaining intervals, it is a continuing process like when a function is ending its surrounding region then other function starts its effectiveness as parameter value changing. In other words, every parameter has at most under the three basis function. The system of equations is easily solved for arbitrary constants $C_i$'s. We have,

$$[C]=[F][R]^{-1} \qquad (20)$$

So the constants $C_i$'s are solved by above equation (5.13). Substituting these constants in equation (2), the approximation solution is obtained. Now the final approximation solution is evaluated at each node (Collocation point). The exact solution also evaluated at these points and result values are compared with each other to find out the accuracy of the NURBS Collocation Method. The effectiveness of present method is studied by considering differential equation of a test problem as follows.

## 4. TEST PROBLEM:

Considering a 1-D heat transfer problem, for conduction through a slab of thickness $L$=0.25m, subjected to inside temperature of $T_1$=110˚C and outside temperature of $T_2$=40˚C, and thermal conductivity of $k$ (W/m.℃) calculating the temperature distribution along the wall thickness at different points (nodes) as shown in figure 1.

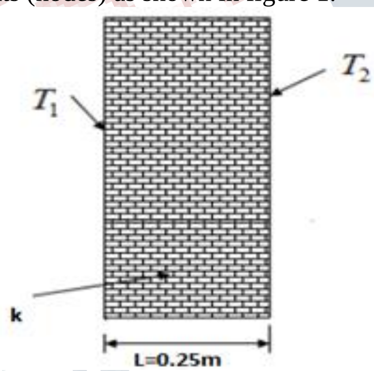


***Figure1. Conduction heat transfer through a slab.***

Governing differential equation is

$$\frac{d^2T}{dx^2}=0 \quad \text{for} \quad 0 \le x \le 0.25 \qquad (21)$$

with boundary conditions $T(0)=T_1$ and $T(L)=T_2$ and having a exact solution

$$T(x)=T_1+\frac{(T_2-T_1)x}{L} \qquad (22)$$

Comparing the given differential equation with equation (1) we have F(x)=0, $a$=0, $b$=0.25 and $d_1=T_1$, $d_2=T_2$, Q(x)= P(x)=0. Taking the approximation function from the equation (2), it can be written as

$$T^h(x)=\sum_{i=-2}^{n-1} C_i R_{i,p}(x)$$

Taking number of intermittent segments (or sub domains) as 11 (i.e. n=11), order of NURBS curve as 3 (i.e. p=3). $X$={a=$X_1$=0, $X_2$, …. $X_{n-1}$, $X_n$=b} with non-uniform values between [a b], for homogeneous coordinates (weights) $h_1$=0.9, $h_i$=1, i≠1and knot vector having 11 elements or knot values. Now the above equation can be modified as

$$T^h(x)=\sum_{i=-2}^{10} C_i R_{i,3}(x) \qquad (23)$$

Substituting the approximation function in governing equation we have

$$\sum_{i=-2}^{10} C_i R''_{i,3}(x)=0 \qquad (24)$$

Knot vector is $x_i$ = {0, 0.0203, 0.0921, 0.1090, 0.1117, 0.1217, 0.1564, 0.1939, 0.1951, 0.2323, 0.25}.

Now let the approximation solution satisfy the boundary conditions. The given boundary conditions are from equation (3)

$$\sum_{i=-2}^{10} C_i R_{i,3}(0)=110 \quad \text{and}$$

$$\sum_{i=-2}^{10} C_i R_{i,3}(0.25)=40 \qquad (25)$$

Rewriting the matrix form of equation (19) using boundary condition equations (25), we get the system of (12) × (12) equations in which (12) arbitrary constants are involved

$$\begin{bmatrix} R_{-2,3}(0) & R_{-1,3}(0) & R_{1,3}(0)\ldots\ldots & R_{10,3}(0) \\ R''_{-2,3}(0) & R''_{-1,3}(0) & R''_{1,3}(0)\ldots\ldots & R''_{10,3}(0) \\ R''_{-2,3}(0.0203) & R''_{-1,3}(0.0203) & R''_{1,3}(0.0203)\ldots\ldots & R''_{10,3}(0.0203) \\ R''_{-2,3}(0.0921) & R''_{-1,3}(0.0921) & R''_{1,3}(0.0921)\ldots\ldots & R''_{10,3}(0.0921) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ R''_{-2,3}(0.2323) & R''_{-1,3}(0.2323) & R''_{1,3}(0.2323)\ldots\ldots\ldots & R''_{10,3}(0.2323) \\ R_{-2,3}(0.25) & R_{-1,3}(0.25) & R_{1,3}(0.25)\ldots\ldots & R_{10,3}(0.25) \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_1 \\ . \\ . \\ . \\ . \\ . \\ C_{10} \end{bmatrix} = \begin{bmatrix} 110 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ 0 \\ 40 \end{bmatrix} \qquad (26)$$

Now the final approximation solution is evaluated at each node (Collocation point) i.e., $x_i$=0, 0.0203, 0.0921, 0.1090 …0.2323 and the values field variable T(x) at each

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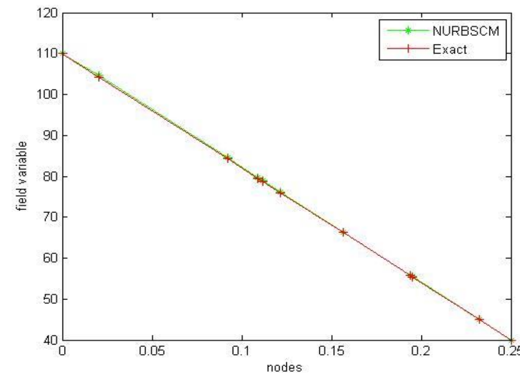
node are calculated. The exact solution also evaluated at these points and result values of field variable T(x) are compared with each other to find out the accuracy of the NURBS Collocation Method and shown in Table 1.

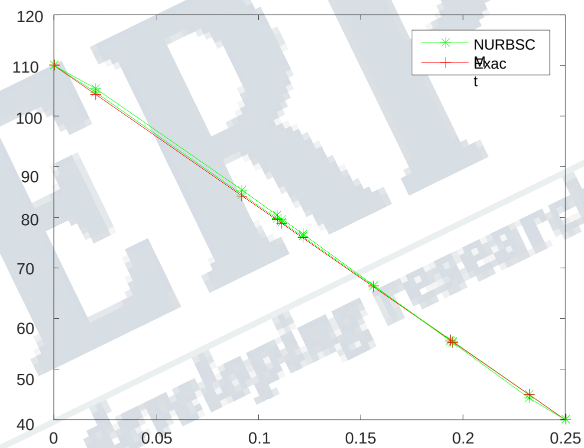
***Table 1: Comparison of field variable (T) with exact solutions for knot vector $x_i$={0,0.0203,......0.2323,0.25} and weights $h_i$={0.9,1,1,....1,1}.***

| Node (knot values) | Exact Solution | NURBS Collocation Method Solution |
|---|---|---|
| 0 | 110 | 110 |
| 0.0203 | 104.3212 | 104.7359 |
| 0.0921 | 84.2061 | 84.4911 |
| 0.1090 | 79.4899 | 79.7445 |
| 0.1117 | 78.7251 | 78.9748 |
| 0.1217 | 75.9246 | 76.1562 |
| 0.1564 | 66.2067 | 66.3757 |
| 0.1939 | 55.7001 | 55.8013 |
| 0.1951 | 55.3841 | 55.4833 |
| 0.2323 | 44.9430 | 44.9749 |
| 0.25 | 40 | 40.0 |

From the above Table 1, it can be stated that the values of the field variable obtained by NURBS Collocation Method, using unequal weights, are nearer to the exact solution values. The absolute % of error at each node is calculated and the maximum absolute % of error is 0.3975. These comparision is given graphically in figure 2. The accuracy of the solution




(a)



(b)

Figure 2: Comparison of field variable T(x) with exact solutions for non-uniform knot spacing, (a) *p*=3 and (b) *p*=4.

### 5. CONCLUSIONS:

In this work, an attempt has been made to use the NURBS basis functions as the basis functions in the collocation method. A non-uniform knot vector was used to obtain the first, second and third degree NURBS basis functions. The derivatives of basis functions were obtained from recursive formula. The method has been applied for solving 1-D conduction heat transfer through slab. The results were compared with exact solution and found that the present method is in good agreement with exact solution.

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