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A Petri Net Model of Friends Sharing Drinks

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***Abstract*— A Petri net is a particular kind of bipartite directed graph. The notion of the Petri net was developed by Carl Adam Petri in 1962. Its application is through modeling. Petri net's theoretical features enable accurate modeling and analysis of system behavior, and its graphical representation makes it possible to visualize the modeled system.**

In this paper, a Petri Net model of four friends eating their respective food items but only having two drinks which will be shared among them has been discussed. Furthermore, the analysis of this model is done by using the reachability tree and matrix equations. Its Safeness, Conservation, Coverability, Liveness, and deadlock have been discussed in detail.

Index Terms- Places, Transitions, Tokens, Enabling Transition, Marking, Reachability tree, Matrix Equations.

I. INTRODUCTION

Petri net was introduced in 1962 by Carl Adam Petri. It is a particular kind of bipartite-directed graph. The application of Petri net is through modeling. Petri net theory allows a system to be modeled by a Petri net which is a mathematical representation of the system.

A Petri net is made up of four components: an input function I, an output function O, a collection of places P, and a set of transitions T. Transitions and places are related via the input and output functions. The input function I is a mapping from a transition t_j, to a collection of places $I(t_j)$, known as the input places of the transition. The output function O maps a transition t_j to a collection of places $O(t_j)$ known as the output places of the transition. That is, Petri net consists of Places, Transitions, Input function, and Output function. Tokens are the most important part of it, they show the movements/changes in the Petri net.

Tokens are created and destroyed in the places (conditions) and can travel in a system under certain parameters that can change the state of the system.

1.1. Definitions[1,4,5]

Petri Net

A Petri Net structure is composed of four components and is written as a four tuple,

PN = (P,T,I,O)

where,
P is the set of places
T is the set of transitions
I is the Input function
O is the Output function

Places

The places refer to a certain set of conditions that are to be satisfied. Places are denoted using circles ○ in a Petri net structure.

Transitions

The transitions are the events or activities that occur and lead to the change in the state of the system. These are denoted using a vertical line | or a rectangular bar.

Directed edges or arcs connect the places and transitions.

Tokens

Present in a system are some basic entities called tokens which get created and destroyed in the places (conditions) and can travel in a system under certain parameters that can change the state of the system.

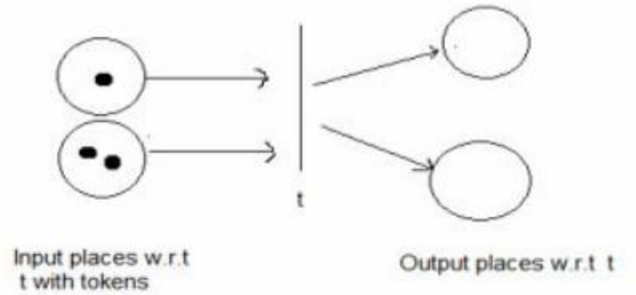

Figure 1. Pictorial representation of Petri net.

With respect to the above Petri Net structure, one can deduce that for the transition t, there exist total four places p_1, p_2, p_3, p_4. Further, it is evident that these places and the transition are connected respectively by arcs or the directed edges. For the input places, p_1 has a single token and p_2 has two tokens.

Enabling Transition

Any transition, say t_j in a system, is enabled and can fire with one or multiple input places; if the number of tokens in all the input places is at least equal to the multiplicity of all the input arcs for t_j of those places respectively. We also call this the triggering of t_j. When t_j in a system triggers, a token gets deleted from its input places and eventually gets created in the respective output places.

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Marking

A marking of a Petri Net is the assignment of the tokens to the set of places. It is denoted by

$M = M_0 . M_1, M_2, \ldots., M_m$ where M_i gives the number of token that are available at the place p_i.

A marking M is a function defined from P, the set of all places to the non-negative integers i.e., $M: P \rightarrow Z^+$. The initial marking is denoted by $M_0: P \rightarrow Z^+$. A marked Petri Net w.r.t M_0 is a 5-tuple structure where

$PN = (P, T, I, O, M_0)$.

It is obvious to realise that the number of token which can be assigned to any place in a Petri net is not bounded, and thus, there are significantly infinite many number of markings possible for the Petri net.

1.2. Analysis of Petri Net

The system modeled through Petri net needs to be analyzed so that some conclusions can be made. Basically, the Petri net is analyzed through

- **Reachability tree** which represents all the reachable markings along with the firing sequence.
- **Matrix equations** The input matrix denoted by D^- is of order $|T| * |P|$ matrix $[a_{ij}]$, where

$$a_{ij} = \begin{cases} 1: p_j \textit{ is an input place for transition } t_i \\ \quad 0: \textit{otherwise} \end{cases}$$

Similarly, the output matrix denoted by D^+ is of order $|T| * |P|$ matrix $[a_{ij}]$, where

$$\boldsymbol{a_{ij}} = \begin{cases} 1: p_j \textit{ is an output place for transition } t_i \\ \quad 0: \textit{otherwise} \end{cases}$$

The incidence matrix denoted by D is defined as

$D = D^+ - D^-$.

II. PETRI NET MODEL OF FOUR FRIENDS SHARING TWO DRINKS

In day to day life, we eat food items along with some shakes/drinks. It happens many times that the same drink is being shared between some of us. In this section, a Petri net model of four friends has been discussed where the four friends are eating their respective food items but only two drinks are shared among them in such a way that the first drink will be shared between two of them while the second drink will be shared among three of them such that only one person will be having both the drinks.

WLOG one can assume that 1^{st} and 2^{nd} person will be sharing first drink while 2^{nd}, 3^{rd} and 4^{th} person will be sharing second drink. Each one of them will be in either eating or drinking state and at a time at most two of them can be in drinking state as when the drink is kept on table then only one can have it. The Petri net corresponding to this situation is:

Depiction of Places (conditions) and Transitions (activities) is as follows:

p_{1e} = First person in the eating state.
p_{1d} = First person in the drinking state.
p_{2e} = Second person in the eating state.
p_{2d1} = Second person having 1^{st} drink.
p_{2d2} = Second person having 2^{nd} drink.
p_{3e}= Third person in the eating state.
p_{3d} = Third person in the drinking state.
p_{4e} = Fourth person in the eating state.
p_{4d} = Fourth person in the drinking state.
p_{d1} = First drink which will be shared by 1^{st} and 2^{nd} person.
p_{d2} = Second drink which will be shared by 2^{nd}, 3^{rd} and 4^{th} person.
t_1 = First person takes the drink.
t_2 = First person puts the drink back on the table.
t_3 = Second person takes the 1^{st} drink.
t_4 = Second person puts the 1^{st} drink back on the table .
t_5 = Second person takes the 2^{nd} drink.
t_6 = Second person puts the 2^{nd} drink back on the table .
t_7 = Third person takes the drink.
t_8 = Third person puts the drink back on the table.
t_9 = Fourth person takes the drink.
t_{10} = Fourth person puts the drink back on the table.

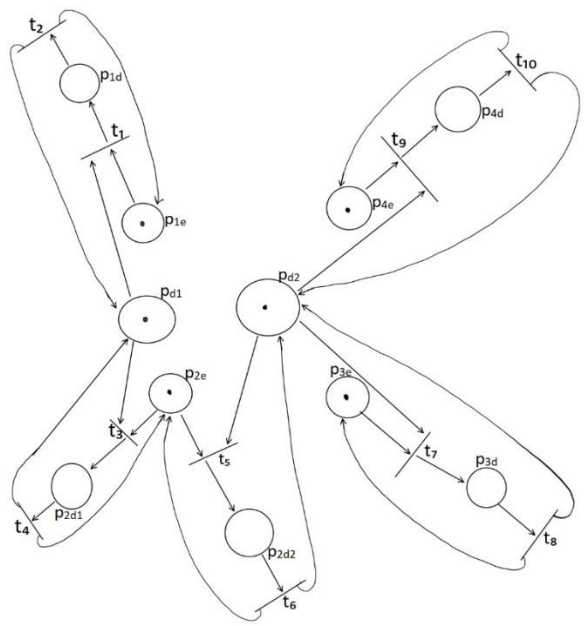

Figure 2. Petri net model of four friends sharing two drinks

Whenever there is a token in p_{d1} or p_{d2} that means the 1^{st} or the 2^{nd} drink is kept on the table, respectively. Tokens in p_{1e}, p_{2e}, p_{3e} and p_{4e} denote that the respective person is eating the food and the token in p_{1d}, p_{2d1} denote that the respective person is having 1^{st} drink while token in p_{2d2}, p_{3d} and p_{4d} denote that the respective person is having 2^{nd} drink.

The analysis of this Petri net structure can be done by both reachability tree and matrix equations.

III. REACHABILITY TREE [2,4,5]

A reachability tree represents all the reachable markings along with the firing sequence. The reachability tree of Petri net in Fig.2 is depicted in Fig.3

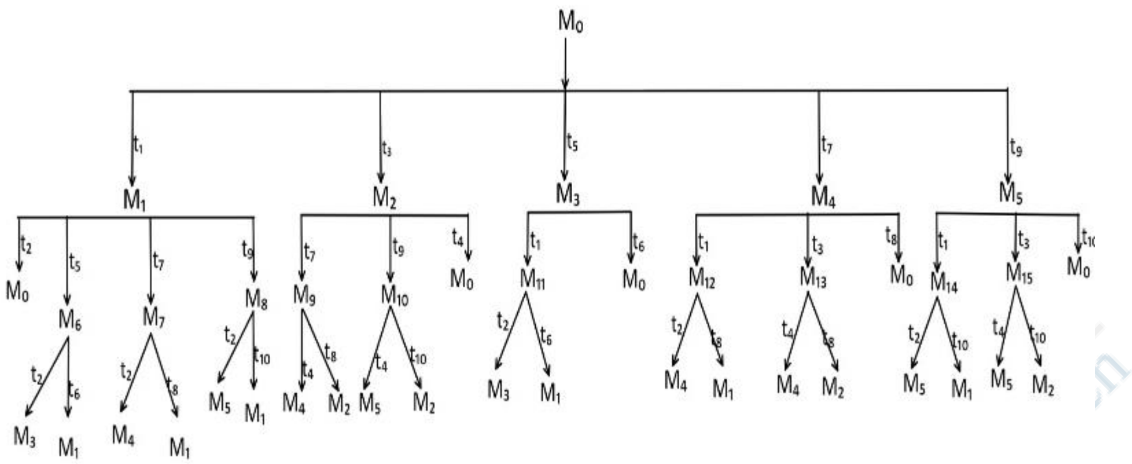

Figure 3. Reachability tree for the Petri net structure in Fig.2

$M_1 = (0, 1, 1, 0, 0, 1, 0, 1, 0, 0, 1)$
$M_2 = (1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1)$
$M_3 = (1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0)$
$M_4 = (1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0)$
$M_5 = (1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0)$
$M_6 = (0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0)$
$M_7 = (0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0)$
$M_8 = (0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0)$
$M_9 = (1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0)$
$M_{10} = (1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0)$
$M_{11} = (0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0)$
$M_{12} = (0, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0)$
$M_{13} = (1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0)$
$M_{14} = (0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0)$
$M_{15} = (1, 0, 0, 1, 0, 1, 0, 0, 1, 0, 0)$

3.1 Analysis of the model using reach ability tree [3,4,5]

i. **Safeness and Boundedness**: If there are never more than one token in a place then that place is safe. A Petri net is safe if every place within it is safe, and a place is k-safe or k-bounded, if the place is k-bounded, which means the number of token in that place cannot exceed a positive integer k, then the petri net is k-safe or k-bounded.

Here, one can clearly see in reachability tree or in all possible markings that all the places have either 0 or 1 token. Hence, this Petri net is safe or 1−bounded.

For example, in M_0 places $p_{1e}, p_{2e}, p_{3e}, p_{4e}, p_{d1}$ and p_{d2} have one token each while all other remaining places have no token.

ii. **Conservation**: A Petri net with initial marking M_0 is strictly conservative if for all M' .

$$\sum_{p_i \in P} M'(p_i) = \sum_{p_i \in P} M_0(p_i)$$

That is, the total number of token in the Petri net at any instant of time must be always equal to number of total token in the initial marking. This Petri net is not strictly conservative as $\sum_{p_i \in P} M_0(p_i) = 6$ while after t_1 fires

$\sum_{p_i \in P} M_1(p_i) = 6$ and we know $5 \neq 6$.

Hence, this Petri net is not strictly conservative.

iii. **Coverability**: All the coverable markings are shown in the reachability tree. The sequence of transitions leading from the initial marking to the covering marking is given by the path from the root to the covering marking. A marking like $(1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0)$ is not in the reachability tree means that **this marking will not be covered or cannot be reached.**

iv. **Liveness**: **A transition is live in marking *M* if it can fire**. A Petri net is said to be live if in all the marking there is atleast one transition which is live i.e. it can fire. In this Petri net, one can easily see in the reachability tree that in all the possible marking, there is always atleast one transition which can fire, so this Petri net structure is live.
For example,
in marking M_0, transitions t_1,, t_3, t_5, t_7 and t_9 can fire,
in marking M_1,, transitions t_2, t_5, t_7 and t_9 can fire,
in marking M_3,, transitions t_1 and t_6 can fire,
in marking M_{12},, transitions t_2 and t_8 can fire.

v. **Deadlock**: A deadlock in a Petri net occurs when no transition can fire in any of the marking. In all the markings of this Petri net, there is always at least one transition which can fire. Hence, no deadlock comes in this Petri net structure.
If a Petri net is live then deadlock cannot occur.

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IV. MATRIX EQUATIONS [2,4]

An alternative way to define a Petri net is to define two matrices D^- and D^+ to represent the input and output functions, respectively.

The input (D^-) and output (D^+) matrices of Fig. 2 will be of order $(10 * 11)$, the two matrices are as follows:

$$D^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$D^+ = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The incidence matrix $D = D^+ - D^-$

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Conservative Behaviour:

"A Petri Net is conservative iff there exists a positive vector w such that $D.w = 0$"

and

"A Petri net is partially conservative iff there exists a non-negative vector w such that $D.w = 0$"

Here, the system of equations w.r.t. the incidence matrix D by taking

$w = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}, w_{11})^T$ are:

$$-w_1 + w_2 - w_{10} = 0$$
$$w_3 - w_4 + w_{10} = 0$$
$$w_3 - w_5 + w_{11} = 0$$
$$w_6 - w_7 + w_{11} = 0$$
$$w_8 - w_9 + w_{11} = 0$$

Solution of these equations gives the weighting vector

$$w = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

which is a non-negative vector. Hence, the Petri net is partially conservative.

To find reachable marking using given firing sequence:

If any marking M' is reachable from any marking M_i then there exist a firing sequence (σ) of transitions. If firing sequence of transitions takes the net from M_i to M then

$$M' = M + f(\sigma).D,$$

where,

$f(\sigma)$ is the non- negative firing vector of the firing sequence.

For example: If there are 4 transitions in the net t_1, t_2, t_3, t_4 and the firing sequence is $\sigma = t_1 t_3$ then the firing vector is $f(\sigma) = (1,0,1,0)$

In the Petri net structure, Fig.0; from the initial marking M_0 using firing sequence, $t_7 t_3$ the reachable marking will be:

$$M' = M + f(\sigma).D$$
$$M' = (1,0,1,0,0,1,0,1,0,1,1) +$$
$$(0,0,0,-1,1,0,-1,1,0,0,-1,-1)$$
$$M' = (1,0,0,1,0,0,1,1,0,0,0) = M_{13}$$

To check whether the given marking is reachable or not?

Let us check that the marking

$M' = (1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0)$ is reachable from initial marking M_0 or not.

Using the formula, $M' = M_0 + f(\sigma).D$, we need to determine the firing sequence to reach this M'

$$M' = M_0 + f(\sigma).D$$
$$(1,0,1,0,1,0,0,1,0,1,0) = (1,0,1,0,0,1,0,1,0,1,1) + x.D$$

Here, no such $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ can be calculated. That means this marking

$M' = (1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0)$ cannot be obtained.

4.1 Analysis using Labeled Place-Transitive Matrix [2]

For studying properties like cyclic/acyclic, conflict, self-loop, we compute Labeled Place-Transitive Matrix.

The Labeled Place-Transitive matrix is defined as:

$L_{BP} = [D^-]^T . D_t . D^+$ of order $|P| * |P|$

where,

D^- is the input matrix

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D^+ is the output matrix

D_t is the diagonal-transition matrix for $T = t_1, t_2, \ldots\ldots, t_n$ here, n = 10.

The Labeled Place-Transitive Matrix is:

$$L_{BP} = \begin{bmatrix} 0 & t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_3 & t_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & t_4 & 0 \\ 0 & 0 & t_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_8 & 0 & 0 & 0 & 0 & t_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{10} & 0 & 0 & t_{10} \\ 0 & t_1 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_5 & 0 & t_7 & 0 & t_9 & 0 & 0 \end{bmatrix}$$

Corresponding row and column transition vectors are:

$$T_R = [t_2\ 2t_1\ t_4 + t_6\ 2t_3\ 2t_5\ t_8\ 2t_7\ t_{10}\ 2t_9\ t_2 + t_4\ t_6 + t_8 + t_{10}]$$

and

$$T_C = [t_1\ 2t_2\ t_3 + t_5\ 2t_4\ 2t_6\ t_7\ 2t_8\ t_9\ 2t_{10}\ t_1 + t_3\ t_5 + t_7 + t_9]^T$$

where, the row-transition vector T_R is the sum of corresponding columns of L_{BP} and the column-transition vector T_C is the sum of corresponding rows of L_{BP}

The components/entries of T_R and T_C are finite linear combinations of transitions with positive integer coefficients. The number of incoming/outgoing arcs from the transitions are represented by the coefficients of transitions that are present in the transition vectors T_C (T_R).

i. Cyclic/Acyclic Nature:

A Petri Net is acyclic if at least one entry in T_R or T_C is zero. The T_R and T_C corresponding to our proposed model has no zero component. Hence, this Petri net is cyclic in nature.

The cycles are computed as:

The transition vectors (row and column) are inputs.

a. Select any i^{th} entry(place) of T_C
b. Select any transition in the i^{th} component of T_C
c. locate this transition in the components of T_R
d. Select an entry(place) in T_R corresponding to the selected transition in step(b)
e. If this selected entry is same as previously selected entry then end the cycle otherwise go back to step(b).

Some of the possible cycles are:

$p_{1e} \rightarrow t_1 \rightarrow p_{1d} \rightarrow t_2 \rightarrow p_{1e}$

$p_{1d} \rightarrow t_2 \rightarrow p_{1e} \rightarrow t_1 \rightarrow p_{1d}$

$p_{2e} \rightarrow t_3 \rightarrow p_{2d1} \rightarrow t_4 \rightarrow p_{2e}$

$p_{d2} \rightarrow t_7 \rightarrow p_{3d} \rightarrow t_8 \rightarrow p_{d2}$

ii. Conflict:

Petri Net is conflict-free if and only if every component of T_C has exactly one transition, or, if any component has more than one transition then the same transitions must appear in the corresponding component of T_R.

Here, 3^{rd} component of T_C has two different transitions that are t_3 and t_5 and the corresponding component of T_R has different transitions implying that there is conflict between t_3 and t_5. Hence, this Petri net is in conflict.

iii. Self-loop(free):

A Petri Net is said to be self loop free if and only if the corresponding components of both T_R and T_C have no transitions that are same or identical at a certain place p_k . In our proposed model, there are no identical corresponding components of T_R and T_C so this Petri net is self-loop free.

iv. Structural Concurrency:

A Petri Net is said to be structurally concurrent iff at least one component of the column transition vector T_C has coefficient 2 or greater than 2. Our Petri net structure is concurrent as in T_C, components corresponding to places $p_{1d}, p_{2d1}, p_{2d2}, p_{3d}, p_{4d}$ has coefficient 2.

V. CONCLUSION

In this paper, the model of four friends sharing two drinks while they have their respective eating items has been designed and analysed. It has been observed by analysing the model that the model is safe, not strictly conservative, live, cyclic, has conflict, self-loop free and is structurally concurrent.

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