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Exact Zero-Divisor Graph

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***Abstract*— the rings considered in this article are commutative rings with non-zero identity which are not integral domains. The aim of this paper is to define and study Exact Zero-Divisor Graphs of a commutative ring with non-zero identity.**

***Index Terms*— Exact Zero Divisors, Exact Zero-Divisor Graph, Zero Divisors**

I. INTRODUCTION

Let R be a commutative ring with nonzero identity. Following [2], we say that an element x of R is exact zero-divisor if there exist $y \in R^*$ such that $Ann(x) = \{r \in R | rx = 0\}$ is a principal ideal yR, whose annihilator is xR, i.e. $Ann(x) = yR \,\&\, Ann(y) = xR$.. We say that $EZ(R)$ is the set of exact zero-divisors of R. We associate a simple graph $E\Gamma(R)$ to R with vertex set is $EZ(R)^* = EZ(R)\{0\}$, the set of nonzero exact zero-divisors of R. Two vertices x and y of are adjacent if and only if (x, y) is a pair of exact zero-divisors, i.e. $Ann(x) = yR \,\&\, Ann(y) = xR$. Clearly, $E\Gamma(R)$ is an empty graph if R is an integral domain.

I. Beck introduced the idea of zero-divisor graph $\Gamma(R)$ in [5]. The definition of zero-divisor graph given by Beck was modified by Anderson and Livingston in [1]. The vertex set of zero-divisor graph $\Gamma(R)$ is the set of nonzero zero divisors of R and two vertices x and y are adjacent if and only if $xy = 0$. Exact zero divisors were introduced by I. B. Henriques & L. M. Sega in [2]. Motivated by the study of zero-divisor graphs in [1], we define and study the exact zero-divisor graphs of commutative rings with non-zero identity. The graphs considered in this article are simple undirected graphs.

In section II, we define and give several examples of exact zero-divisor graphs. In section III, we discuss some properties of exact zero-divisor graphs and compare with properties of zero-divisor graphs. We will include basic definitions of graph theory as needed.

Throughout the article, all ring considered are commutative rings with non-zero identity. For a subset $A \subseteq R$, $A^* = A\{0\}$,. To avoid trivialities we assume that R is not an integral domain when necessary. We follow [4] for standard notations.

II. DEFINITIONS & EXAMPLES

In this section, we define exact zero-divisors and exact zero-divisor graphs, and we give several examples of it. As mentioned in introduction, the rings are commutative rings with non-zero identity which are not integral domains unless otherwise stated.

Definition 2.1 An element x of R is exact zero-divisor if there exist $y \in R^*$ such that $Ann(x) = \{r \in R | rx = 0\}$ is a principal ideal yR, whose annihilator is xR, i.e. $Ann(x) = yR \,\&\, Ann(y) = xR$.
In this case, we say that (x, y) is a pair of exact zero divisors.

Definition 2.2 Let $EZ(R)^*$ be the set of non-zero exact zero divisors of R. We associate a simple graph $E\Gamma(R)$ to R with vertex set $EZ(R)^*$ and two vertices x and y are adjacent if and only if (x, y) is a pair of exact zero-divisors, i.e. $Ann(x) = yR \,\&\, Ann(y) = xR$.

Clearly $E\Gamma(R)$ is an empty graph if R is an integral domain. If $R = A \times B$ where $A \,\&\, B$ are integral domains. Then $Z(R) = X \cup Y$, where $X = \{(x, 0) | x \in A\}$ and $Y = \{(0, y) | y \in B\}$. Zero ideal is an annihilator for all elements $(x, y) \in R$. But we exclude zero ideal as we consider only nonzero exact zero-divisors to be vertices in $E\Gamma(R)$.

Example 2.3 Let $R = Z_3 \times Z_3$.
Then $Z(R)^* = \{(1,0), (2,0), (0,1), (0,2)\}$.
$Ann\big((1,0)\big) = \{(0,1), (0,2)\} = (0,1)R = (0,2)R$
$Ann\big((2,0)\big) = \{(0,1), (0,2)\} = (0,1)R = (0,2)R$
$Ann((0,1)) = \{(1,0), (2,0)\} = (1,0)R = (2,0)R$
$Ann((0,2)) = \{(1,0), (2,0)\} = (1,0)R = (2,0)R$
Thus $E\Gamma(R)$ is as in figure 1.

Example 2.4 Let $R = Z_3 \times Z_5$.
Then $Z(R)^* = \{(1,0),(2,0),(0,1),(0,2),(0,3),(0,4)\}$.
$Ann((1,0)) = \{(0,1),(0,2)\} = (0,1)R = (0,2)R = (0,3)R = (0,4)R$
$Ann((2,0)) = \{(0,1),(0,2)\} = (0,1)R = (0,2)R = (0,3)R = (0,4)R$
$Ann((0,1)) = \{(1,0),(2,0)\} = (1,0)R = (2,0)R$
$Ann\big((0,2)\big) = \{(1,0),(2,0)\} = (1,0)R = (2,0)R$
$Ann((0,3)) = \{(1,0),(2,0)\} = (1,0)R = (2,0)R$
$Ann((0,4)) = \{(1,0),(2,0)\} = (1,0)R = (2,0)R$
$\therefore EZ(R)^* = \{(1,0),(2,0),(0,1),(0,2),(0,3),(0,4)\}$
Thus $E\Gamma(R)$ is as in figure 2.

Example 2.5 Let $R = Z_2[X]/(X^4)$. Let $Im(X) = \bar{x}$.

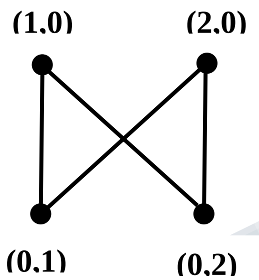


Figure - 1

Then
$Z(R)^* = \{\bar{x}, \overline{x^2}, \overline{x^3}, \overline{x+x^2}, \overline{x+x^3}, \overline{x^2+x^3}, \overline{x+x^2+x^3}\}$.
$Ann(\bar{x}) = \overline{x^3}R$ & $Ann\big(\overline{x^3}\big) = \bar{x}R$
$Ann\big(\overline{x^2}\big) = \overline{x^2}R$
Thus $EZ(R)^* = \{\bar{x}, \overline{x^2}, \overline{x^3}\}$. Thus, $E\Gamma(R)$ is as in figure 3.

Example 2.6 Let $R = Z_4$.
Then $Z(R)^* = \{\bar{2}\}$.
$Ann(\bar{2}) = \bar{2}R$.
Thus $E\Gamma(R)$ is a single vertex as in figure 4.

Example 2.7 Let $R = Z_2[X]/(X^2)$. Let $Im(X) = \bar{x}$.
Then $Z(R)^* = \{\bar{x}\}$.
$Ann(\bar{x}) = \bar{x}R$.
Thus $E\Gamma(R)$ is a single vertex as in figure 5.

Example 2.8 Let $R = Z_2[X,Y]/(X^2, XY, Y^2)$.
Then $Z(R)^* = \{\bar{x}, \bar{y}, \overline{x+y}\}$. (Here, $Im(X) = \bar{x}$).
In this case $E\Gamma(R)$ is an empty graph.

Example 2.9 Let $R = Z_9$.
Then $Z(R)^* = \{\bar{3}, \bar{6}\}$.
$Ann(\bar{3}) = \bar{6}R$
$Ann(\bar{6}) = \bar{3}R$
Thus $E\Gamma(R)$ is as in figure 6.

Example 2.10 Let $R = Z_6$.
Then $Z(R)^* = \{\overline{2,3,4}\}$.
$Ann(\bar{2}) = \bar{3}R$
$Ann(\bar{3}) = \bar{2}R = \bar{4}R$
$Ann(\bar{4}) = \bar{3}R$.
Thus $E\Gamma(R)$ is as in figure 7.

III. PROPERTIES OF $E\Gamma(R)$

In this section we will discuss some properties of $E\Gamma(R)$ and compare with the properties of $\Gamma(R)$.

Remark 3.1 In [1], theorem 2.3, it is shown that the zero-divisor graph $\Gamma(R)$ is connected for any commutative ring R. Example 2.5 indicates that exact zero-divisor graphs need not be connected.

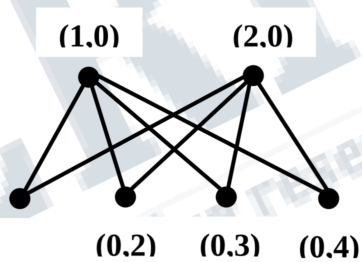


Figure - 2

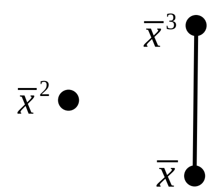


Figure - 3

Remark 3.2 In [1], theorem 2.2, it is shown that the zero-divisor graph $\Gamma(R)$ is finite if and only if R is finite or an integral domain. The result is not true in case of exact zero-divisor graph. For instance, let $R = Z \times Z$. Then $Z(R) = \{X \cup Y\}$; where $X = \{(x,0) | x \in Z\}$, $Y = \{(0,y) | y \in Z\}$. But the vertex set of $E\Gamma(R)$ is $\{(1,0),(-1,0),(0,1),(0,-1)\}$, and the exact zero-divisor graph $E\Gamma(R)$ of R is as in figure 8.

It is known that a graph Γ is *connected* if there is a path between any two distinct vertices. The length of shortest path between two distinct vertices x and y is denoted by d(x, y) and d(x, y) = ∞ if no such path exists. The *diameter* of a graph Γ is denoted and defined as $diam(\Gamma) = sup\{d(x,y) | x \,\&\, y \text{ are distinct vertices of } \Gamma\}$.

The girth of a graph Γ is denoted by $g(\Gamma)$ and defined to be length of the shortest cycle in Γ. $g(\Gamma) = \infty$ if Γ contains no cycle.

Theorem 3.3 If $E\Gamma(R)$ is connected, then the shortest path between any two distinct vertices of $E\Gamma(R)$ is at most 2.

Proof: Let $E\Gamma(R)$ is connected. Suppose the length of shortest path between any two vertices is not two. So let $x - x_1 - x_2 - y$ be a path of shortest length three between two vertices $x \,\&\, y$ of $E\Gamma(R)$. By the definition of $E\Gamma(R)$, we have $Ann(x) = x_1R \,\&\, Ann(x_1) = xR$. Similarly $Ann(x_1) = x_2R \,\&\, Ann(x_2) = x_1R$. $Ann(x_2) = yR \,\&\, Ann(y) = x_2R$. But then $Ann(x) = x_1R = Ann(x_2) = yR$ and $Ann(y) = x_2R = Ann(x_1) = xR$. Thus $x - y$ are adjacent in $E\Gamma(R)$. Hence the shortest path between any two vertices of $E\Gamma(R)$ cannot exceed two. □

Theorem 3.4 Let R be a commutative ring with unity. If the exact zero-divisor graph $E\Gamma(R)$ of R is connected, and contains a cycle, then $g(E\Gamma(R)) \leq 4$.

Figure - 4

Figure - 5

Figure - 6

Proof: From theorem 3.3, we can see that the length of shortest path between any two distinct vertices is at most two. If there is a path of length three between any two distinct vertices, then they are adjacent and we get a cycle of length four. Hence $g(E\Gamma(R)) \leq 4$. □

Theorem 3.5 Let R be a ring of the form $F_1 \times F_2$, where $F_1 \,\&\, F_2$ are fields. Then $E\Gamma(R)$ is connected. Moreover $E\Gamma(R)$ is complete bipartite graph.

Proof: Let $R = F_1 \times F_2$, where $F_1 \,\&\, F_2$ are fields. Hence $Z(R)^* = \{X \cup Y\}$ where $X = \{(\alpha, 0) | \alpha \in F_1^*\}$, $Y = \{(0, \beta) | \beta \in F_2^*\}$. Now $Ann((\alpha, 0)) = \{0\} \cup F_2^*$ & $Ann((0, \beta)) = F_1^* \cup \{0\}$. Therefore $(\alpha, 0)$ is adjacent to each $(0, \beta)$. Hence $E\Gamma(R)$ is connected and complete bipartite graph. □

Remark 3.6 The converse of above theorem is not true. It can be seen from the example discussed in remark 3.2.

Remark 3.7 If $R = F_1 \times F_2 \times F_3$, where F_1, $F_2 \,\&\, F_3$ are fields. Then $E\Gamma(R)$ need not be connected. For instance, let $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. Suppose that there is a path between $e_1 \,\&\, e_2$. Then the length of the shortest path cannot exceed two. Let $e_1 - x - e_2$ be such path, where $x = (x_1, x_2, x_3)$. Now by definition of $E\Gamma(R)$, $\{0\} \times F_2 \times F_3 = Ann(e_1) = (x_1, x_2, x_3)R$. Hence $x_1 = 0$. Similarly, $F_1 \times \{0\} \times F_3 = Ann(e_1) = (x_1, x_2, x_3)R$, hence $x_2 = 0$. Therefore $(x_1, x_2, x_3) = (0, 0, z)$ for some $z \in F_3$. Now, $F_1 \times F_2 \times \{0\} = Ann(0,0,z) = (e_1, 0, 0)R = F_1 \times \{0\} \times \{0\}$. Therefore $F_2 = \{0\}$, contradiction. Hence there is no path between $e_1 \,\&\, e_2$. □

Remark 3.8 If $R = F_1 \times F_2$, where $F_1 \,\&\, F_2$ are fields. From the proof of theorem 3.5, we observe that $(\alpha, 0)$ is adjacent to each $(0, \beta)$, where $\alpha \in F_1^*$, $\beta \in F_2^*$. Hence zero-divisor graph $\Gamma(R)$ and exact zero-divisor graph $E\Gamma(R)$ of R coincide in this case.

Remark 3.9 The exact zero-divisor graph $E\Gamma(R)$ of R is not connected even if R is Artinian ring. (see example 2.5)

We end the article with the following question.

Question: Can we have a characterization for $E\Gamma(R)$ to be connected for a commutative ring R with $1 \neq 0$?

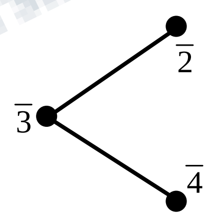
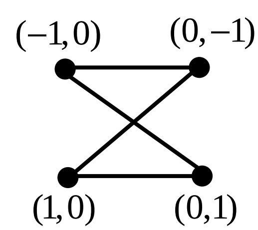


Figure - 7



Figures – 8

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