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Coupled Fixed Point Results for a Contractive Condition in Ordered Partial Metric Spaces

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Abstract: — **In the present paper, we establish some coupled fixed point theorems for a mapping having mixed monotone property satisfying a contractive condition in partially metric spaces. An example is given to show that our results are proper generalizations of the existing ones.**

Keywords: -- **Coupled Fixed Point, Partial Ordered, Complete Metric Space.**

I. INTRODUCTION

The banach fixed point theorem is probably the most well-known fixed point theorem and has been generalized in several directions by many mathematicians. In 1994 Matthews introduced a new notion of generalized metric space called partial metric space [7] In which the distance of appoint from itself may not fixed point from itself may not be zero. The first result on existence of fixed points in partially ordered sets was given by ran and Reurings [11] who presented their applications to a matrix equation. Many authors obtained important results on partial metric spaces.

Recently, Bhaskar and Lakshmikantham [3] introduced the concept of coupled fixed point and the mixed monotone property. Furthermore, they proved some coupled fixed point theorems for mappings which satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for periodic boundary value problem. Ciric and Lakshmikantham [5] Later on investigated some more cupled fixed point theorems in partially ordered sets. Many results on coupled fixed points theorems have been extended to partial ordered metric spaces, EG, In [1], [2], [4], [6], [8], [9], [10].

In this paper, we establish some coupled fixed point theorems in partially metric spaces. We study the necessary condition for the uniqueness of coupled fixed point of the given mapping in partially ordered metric space. We also give example to illustrate our main theorem.

II. PRELIMINARIES

In this section, we give some definitions, lemma which is useful for main result in this paper.**Definition 2.1: [3], [5]**An element $(x,y) \in X \times X$ is said to be coupled fixed point of the mapping $F: X \times X \to X$ if $F(x,\ y) = x, F(y,x) = y$.

Definition 2.2:[3] Let $(X,\ \leq)$ be a partially ordered set and $F: X \times X \to X$. We say that F has the mixed monotone property if $F(x,\ y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$x_1,

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 2.3: [7], [9], [10] Let Xbe a non-empty set. A partial metric on $X$is a function $p: X \times X \to R_n$such that for all $x, y, z \in X$:

(P$_1$) $x = y$ iff $p(x,x) = p(x,\ y) = p(y,y)$,

(P$_2$)$p(x,x) \leq p(x,\ y)$,

(P$_3$) $p(x,\ y) = p(y,x)$,

(P$_4$) $p(x,\ y) \leq \mathrm{p(x,z)+p(z,y)-p(z,z)}$.

A partial metric space is a pair (X,p)such that X is a non- empty set and p is a partial metric on X.

If $p$is a partial metric on X, then the function $p: X \times X \to R_+$given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition2.4: [7], [9], [10] Let (X,p) be a partial metric space. Then:

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(a) A sequence $\{x_n\}$ in partial metric space (X,p)converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$;
(b) A sequence $\{x_n\}$ in partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x_n) =$ lim𝑛→∞𝑝𝑥, 𝑥𝑛; if and only if lim𝑛→∞𝑝𝑠𝑥, 𝑥𝑛=0;
(c) A sequence $\{x_n\}$ in partial metric space (X,p) is called a cauchy sequence if there exists(and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$;
(d) A partial metric space (X,p) is said to be complete if every cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ that is $(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Lemma 2.5: [7], [9] Let (X,p) be partial metric space;
(a) $\{x_n\}$ is Cauchy sequence in (X,p) if and only if it is Cauchy sequence in the metric space$(X$, 𝑝𝑠;
(b) A partial metric space (X,p) is complete if and only if the metric space $(X,\ p^s)$ is complete; furthermore, $\lim_{n\to\infty} p^s(x_n,\ x) = 0iffp(x,x) = \lim_{n\to\infty} p(x_n,\ x) =$ lim𝑛→∞𝑝𝑥𝑛, 𝑥𝑚.

Theorem 2.6: [2] Let $(X,\ \leq)$be a partially ordered set and let $p$be a partial metric on Xsuch that $(X,p)$is complete. Let $f: X \to X$be a non-decreasing map with respectto$\leq$. Suppose that the following conditions hold for$y \leq x$, we have

$$(i)\qquad p(fx,fy) \leq p(x,y) - \psi\big(p(x,y)\big),$$

where$\psi: [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function such that it is positive in$(0,\infty)$, $\psi(0) = 0$ and$\lim_{n\to\infty} \psi(t) = \infty$;
ii) $\exists\, x_0 \in X$ such that$x_0 \leq f x_0$,
iii) f is continuous in (X,p), or
iv) if a non-decreasing sequence $\{x_n\}$converges to $x \in X$,then $x_n \leq x$for all n.
Then $f$has fixed point$u \in X$. Moreover $p(u,u) = 0$.

III. RESULTS AND DISCUSSION

Theorem 3.1: Let $(X,\ \leq)$be a partially ordered set and let $p$be a partial metric on Xsuch that (X,p) is complete. Suppose the mapping $F: X \times X \to X$satisfies the following condition for all $, y, u, v \in X$, we have

$$p\big(F(x,y),F(u,v)\big) \leq p(x,u) - \psi\big(p(y,v)\big) \qquad (1)$$

where ψ:[0, $\infty) \to [0,\infty)$is a continuous and non-decreasing function such that it is positive in$(0,\infty)$, $\psi(0) = 0$and$\lim_{n\to\infty}\psi(t) = \infty$;
(i) F is continuous or
(ii) Xhas the following properties :
(a) if $\{x_n\}$a non-decreasing sequence with $x_n \to x$ then $x_n \leq x$, for all $n \in N$,
(b) if $\{y_n\}$a non-increasing sequence with $y_n \to y$ then $y_n \geq y$, for all $n \in N$,
(iii) $\exists\, x_0, y_0 \in X$ Such that $x_{0,} \leq F(x_0,y_0)$ and$y_0 \geq F(y_0,x_0)$.
Then F has a coupled fixed point$(u^*,v^*) \in X \times X$.

Proof: Since $x_{0,} \leq F(x_0,y_0) = x_1$ (say); $y_0 \geq F(y_0,x_0) = y_1$ (say).
Letting $x_2 = F(x_1,y_1)$and$y_2 = F(y_1,x_1)$;
We denote $F^2(x_0,y_0) = F\big(F(x_0,y_0),F(y_0,x_0)\big) = F(x_1,y_1) = x_2$
Similarly,$F^2(y_0,x_0) = F\big(F(y_0,x_0),F(x_0,y_0)\big) = F(y_1,x_1) = y_2$.
With this notation, we have,
$x_2 = F^2(x_0,y_0) = F(x_1,y_1) \geq F(x_0,y_0) = x_1$,
$y_2 = F^2(y_0,x_0) = F(y_1,x_1) \leq F(y_0,x_0) = y_1$.
Further for $n = 1, 2, 3, \dots\dots\dots$; we get
$x_{n+1} = F^{n+1}(x_0,y_0) = F\big(F^n(x_0,y_0),F^n(y_0,x_0)\big)$,
$y_{n+1} = F^{n+1}(y_0,x_0) = F\big(F^n(y_0,x_0),F^n(x_0,y_0)\big)$.
We can easily verify that
$x_{0,} \leq F(x_0,y_0) = x_1 \leq F^2(x_0,y_0) = x_2 \leq \cdots \dots \dots \leq F^{n+1}(x_0,y_0)$,
$y_0 \geq F(y_0,x_0) = y_1 \geq F^2(y_0,x_0) = y_2 \geq \cdots \dots \dots \geq F^{n+1}(y_0,x_0)$.
We have,

$$p(x_{n+1},x_n) = p\big(F(x_n,y_n),F(x_{n-1},y_{n-1})\big) \leq p(x_n,x_{n-1}) - \psi\big(p(y_n,y_{n-1})\big) \qquad (2)$$

and similarly,

$$p(y_{n+1},y_n) = p\big(F(y_n,x_n),F(y_{n-1},x_{n-1})\big)$$

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$$\leq p(y_n, y_{n-1}) - \psi(p(x_n, x_{n-1})). \quad (3)$$

Therefore, by letting:

$$p_n = p(x_n, x_{n-1}) + p(y_n, y_{n-1}).$$

By adding, we have

$$p_n \leq p_{n-1} - \psi(p_{n-1}). \quad (4)$$

If there exist $n_1 \in N^* p(x_{n_1}, x_{n_1-1}) = 0$, $p(y_{n_1}, y_{n_1-1}) = 0$, then $x_{n_1-1} = x_{n_1} = F(x_{n_1-1}, y_{n_1-1})$, $y_{n_1-1} = y_{n_1} = F(y_{n_1-1}, x_{n_1-1})$ and x_{n_1-1}, y_{n_1-1} is fixed point of F and the proof is finished. In other case $p(x_{n+1}, x_n) \neq 0$; $p(y_{n+1}, y_n) \neq 0$ for all $n \in N$. Then by using assumption on ψ , we have,

$p_n \leq p_{n-1} - \psi(p_{n-1}) \leq p_{n-1}$ (5) $\{p_n\}$ is a non - negative sequence and hence possess a limit p^*. Taking limit when $n \to \infty$, we get,

$$p^* \leq p^* - \psi(p^*)$$

and consequently $\psi(p^*) = 0$. By our assumption on ψ, we conclude $p^* = 0$, ie. $\lim_{n\to\infty} p_n = 0$.

$\lim_{n\to\infty} p(x_{n+1}, x_n) + p(y_{n+1}, y_n) = 0$,

$\Rightarrow \lim_{n\to\infty} p(x_{n+1}, x_n) = \lim_{n\to\infty} (y_{n+1}, y_n) = 0.$

We shall show that $\{x_n\}$, $\{y_n\}$ are Cauchy sequences in X and there exist $x^*, y^* \in X$. Suppose that at least one $\{x_n\}$ or $\{y_n\}$ be not a cauchy sequence. Then there exist $\varepsilon > 0$ and two subsequence of integers n_k, m_k with $n_k > m_k \geq k$, such that

$r_k = p(x_{m_k}, x_{n_k}) + p(y_{m_k}, y_{n_k}) \geq \varepsilon, \forall k =$ 1,2,3 ....Further, corresponding to m_k, we can choose n_k in such a way that it is smallest integer with $n_k > m_k \geq k$ satisfying (7), we have

$p(x_{m_k}, x_{n_{k-1}}) + p(y_{m_k}, y_{n_{k-1}}) < \varepsilon.$

Using (7) and (8) and triangle inequality, we get

$$\varepsilon \leq r_k = p(x_{m_k}, x_{n_k}) + p(y_{m_k}, y_{n_k})$$
$$\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) + p(y_{m_k}, y_{n_{k-1}}) + p(y_{n_{k-1}}, y_{n_k})$$
$$= p(x_{m_k}, x_{n_{k-1}}) + p(y_{m_k}, y_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) + p(y_{n_{k-1}}, y_{n_k})$$

$< \varepsilon + p_{n_{k-1}}$.

Letting $k \to \infty$ and using (6), we have

$\lim_{n,m\to\infty} r_k = \varepsilon > 0.$

Now, we get

$$p(x_{m_{k+1}}, x_{n_{k+1}}) = p(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k})) = p(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k}))$$
$$\leq p(x_{n_k}, x_{m_k}) - \psi\left(p(y_{n_k}, y_{m_k})\right). \quad (10)$$

Similarly,

$$p(y_{m_{k+1}}, y_{n_{k+1}}) = p(F(y_{m_k}, x_{m_k}), F(y_{n_k}, x_{n_k})) = p(F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k}))$$
$$\leq p(y_{n_k}, y_{m_k}) - \psi\left(p(x_{n_k}, x_{m_k})\right). \quad (11)$$

Using (10) and (11), we get

$$r_{k+1} \leq r_k - \psi(r_k) \quad (12)$$

$\forall k \in 1,2,3 \ldots \ldots$....taking $k \to \infty$ on both sides of equation (12)

$\varepsilon \lim_{k\to\infty} r_{k+1} \leq \lim_{k\to\infty} r_k - \psi(r_k) < \varepsilon.$

Which is a contraction. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

By lemma (2.5), $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in (X, p^s). Since (X, p) is complete, hence (X, p^s) is also complete, so there exist $u^*, v^* \in X$ such that,

$$\lim_{n\to\infty} p^s(x_n, u^*) = \lim_{n\to\infty} p^s(y_n, v^*) = 0.$$

By lemma, we get

$p(u^*, u^*) = \lim_{n\to\infty} p(x_n, u^*) = \lim_{n\to\infty} p(x_n, x_n);$

$p(v^*, v^*) = \lim_{n\to\infty} p(y_n, v^*) = \lim_{n\to\infty} p(y_n, y_n).$

By condition and equation, we get

$\lim_{n\to\infty} p(x_n, x_n) = 0.$

It follows that

$p(u^*, u^*) = \lim_{n\to\infty} p(x_n, u^*) = \lim_{n\to\infty} p(x_n, x_n) = 0,$

Similarly

$p(v^*, v^*) = \lim_{n\to\infty} p(y_n, v^*) = \lim_{n\to\infty} p(y_n, y_n) = 0.$

We now prove that $F(u^*, v^*) = u^*$, $F(v^*, u^*) = v^*$. We shall distinguish the cases (i), ii(a) and ii(b) of the Theorem 3.1.

Since X is a complete metric space, there exist $u^*, v^* \in X$ such that $\lim_{n\to\infty} x_n = u^*$, $\lim_{n\to\infty} y_n = v^*$. We now show that if the assumption (i) holds, then (u^*, v^*) is coupled fixed point of F.

As, we have

$$u^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = F(u^*, v^*)$$

and

$v^* = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F\left(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\right) = F(v^*, u^*).$

Therefore, (u^*, v^*) is coupled fixed point of F. Suppose now that the condition ii (a) and ii (b) of the theorem holds.

The sequence $\{x_n\} \to u^*$, $\{y_n\} \to v^*$.

$$p(F(u^*,v^*),u^*) \le p(F(u^*,v^*),x_{n+1}) + p(x_{n+1},u^*) = p\big(F(u^*,v^*), \quad F(x_n,y_n)\big) + p(x_{n+1},u^*) \le p(u^*,x_n) - \psi\big(p(v^*,y_n)\big) + p(x_{n+1},u^*)$$

Letting $n \to \infty$, we have

$$p(F(x,y),x) \le 0 - \psi(0) = 0.$$

This implies that $F(u^*,v^*) = u^*$, similarly, we can show that $F(v^*,u^*) = v^*$. This completes the theorem.

Theorem 3.2: Let the hypotheses of Theorem 3.1 hold. We obtain the uniqueness of the coupled fixed point of F.

Proof: Suppose (u^*,v^*) and (u',v') are coupled fixed points of F, that is, $F(u^*,v^*) = u^*, F(v^*,u^*) = v^*$, $F(u',v') = u'$ and $F(v',u') = v'$. We shall prove that $u^* = u', v^* = v'$.

$(u,v) \in X \times X$By assumption there exist $(u,v) \in X \times X$comparable with both of them.

We define sequences$\{u_n\}$, $\{v_n\}$as follows

$u_0 = u, \quad v_0 = v, \quad u_{n+1} = F(u_n,v_n)$ and $v_{n+1} = F(v_n,u_n)$ $\forall n$. Since (u,v) is comparable with(u^*,v^*), we may assume that

$(u^*,v^*) \ge (u,v) = (u_0,v_0)$.

By using the mathematical induction, it is easy to prove that

$$(u^*,v^*) \ \ge \ (u_n,v_n)\ \forall n. \qquad (13)$$

From (1) and (13), we have

$$p(u^*, \quad u_{n+1}) = p(F(u^*,v^*),F(u_n,v_n)) \le p(u^*,u_n) - \psi(p(v^*,v_n)) \qquad (14)$$

Similarly, we also have

$$p(v^*,v_{n+1}) = p(F(v^*,u^*),F(v_n,u_n)) \le p(v^*,v_n) - \psi(p(u^*,u_n)) \qquad (15)$$

Adding (14) and (15), we get

$p(u^*,\ u_{n+1}) +\ p(v^*,\ v_{n+1}) \le p(u^*,u_n) + p(v^*,v_n) - \psi(p(v^*,v_n) + p(u^*,u_n))$.

This implies

$$p(u^*, \quad u_{n+1}) +\ p(v^*, \quad v_{n+1}) \le p(u^*,u_n) + p(v^*,v_n). \qquad (16)$$

That is, the sequence is decreasing. Therefore there exist $\alpha \ge 0$. Such that

$$\lim_{n\to\infty} p(u^*,u_n)\ + p(v^*,v_n)\ = \alpha\ . \qquad (17)$$

We shall show that $\alpha = 0$. Suppose to contrary, that $\alpha > 0$. Taking the limit as $n \to \infty$ in equation (17), we get

$$\alpha \le \alpha - \psi(\alpha) < \alpha \qquad (18)$$

a contradiction. Thus $\alpha = 0$, that is,

$$\lim_{n\to\infty} p(u^*,u_n)\ + p(v^*,v_n)\ = 0\ . \qquad (19)$$

It implies

$$\lim_{n\to\infty} p(u^*,u_n) = \lim_{n\to\infty} p(v^*,v_n)\ = 0\ . \qquad (20)$$

Similarly we can show that

$$\lim_{n\to\infty} p(u',u_n) = \lim_{n\to\infty} p(v',v_n)\ = 0\ . \qquad (21)$$

From (20) and (21), we have,$u^* = u', v^* = v'$.

Example 3.3: Let $X = [0,1]$endowed with the usual partial metric $p$defined by $p: X \times X \to [0,1]$ with $p(x,y) = \max(x,y)$. The partial metric space $(X,p)$is complete because $(X,\ p^s)$is complete for any$x,y,\in X$,

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2\max(x,y) - (x+y) = |x-y|$$

Thus $(X,\ p^s)$is Euclidean metric space which is complete.

Consider the mapping $F: X \times X \to X$defined by $(x,y) = \frac{x-y}{4}\ ; x \ge y$. Let us take $\psi: [0,\infty) \to [0,\infty)$ such that $\psi(t) = \frac{t}{4}$.

Clearly F is continuous and has the mixed monotone property. Also there are$x_0 = 0, y_0 = 0$in Xsuch that$x_0 = 0 \le F(0,0) = F(x_0,y_0)$ and $y_0 = 0 \ge F(0,0) = F(y_0,x_0)$.

Then it is obvious that $(0,0)$ is the coupled fixed point of F.

Now, we have following possibilities for values of (x,y) and (u,v) such that $x \le u$, $y \ge v$.

Case 1- If $(x,y) = (u,v) = (0,0)$

Then clearly $p\big(F(x,y),F(u,v)\big) = 0$

Thus(1) holds.

Case 2- If $(x,y) = (u,v) = (1,0)$.

Then LHS of (1)

$$= p\big(F(x,y),F(u,v)\big) = p\big(F(1,0),F(1,0)\big) = p\left(\frac{1}{4},\frac{1}{4}\right) = \frac{1}{4}$$

and RHS of (1)

$$= p(1,1) - \psi\big(p(0,0)\big) = 1 - 0 = 1$$

Thus (1) holds.

Case 3- If $(x,y) = (u,v) = (1,1)$.

Then LHS of (1)

$$= p\big(F(x,y),F(u,v)\big) = p\big(F(1,1),F(1,1)\big) = 0$$

and RHS of (1)

$$= p(1,1) - \psi\big(p(1, \quad 1)\big) = 1 - \frac{1}{4} = \frac{3}{4}$$

Thus (1) holds.

Case 4- If $(x, y) = (1, 0);\ (u, v) = (0, 0)$.
Then LHS of (1)

$$= p\big(F(x,y), F(u,v)\big) = p\big(F(1,0), F(0,0)\big)$$
$$= p\left(\frac{1}{4}, 0\right) = \frac{1}{4}$$

and RHS of (1)

$$= p(1,0) - \psi\big(p(0, \quad 0)\big) = 1 - 0 = 1$$

Thus (1) holds.
Case5- If $(x, y) = (1, 0);\ (u, v) = (1,1)$.
Then LHS of(1)

$$= p\big(F(x,y), F(u,v)\big) = p\big(F(1,0), F(1,1)\big)$$
$$= p\left(\frac{1}{4}, 0\right) = \frac{1}{4}$$

and RHS of (1)

$$= p(1,1) - \psi\big(p(0, \quad 1)\big) = 1 - \frac{1}{4} = \frac{3}{4}$$

Thus (1) holds.
Thus all the conditions of theorem 3.1 are satisfied.
Therefore F has a coupled fixed point in X.

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