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Subclass of Harmonic Univalent Functions Associated With the Differential Operator

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Abstract: **In the present paper, we introduce a new subclass of harmonic univalent functions in the unit disk $U=\{z\in C:|z|<1\}$ by using a differential operator. Also we obtain the coefficient bounds, convolution conditions, convex combinations and extreme points.**

***Keywords*: Harmonic, univalent functions, Differential operator. 2010AMS classification: 30C 45, 31 A 05**

I. INTRODUCTION

A continuous complex valued function $f=u+iv$ defined in a simply connected domain D is said to be harmonic in D, if both u and υ are real harmonic in D. In any simply connected domain we can write $f=h+\overline{g}$, where h and g are analytic in D. we call 'h' the analytic part and 'g' the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)|>|g'(z)|$ in D, see[1].

In 1984, Clunie and Sheil-Small [1] investigated the class S_H and studied some coefficient bounds. Since then, there have been several papers published related to S_H and its subclasses. In fact by introducing new subclasses Sheil-Small [18], Silverman [9], Jahangiri[15], Silverman and Silvia [10] and Ahuja[2] presented a systematic and unified study of harmonic univalent functions. Furthermore we refer to Ponnusamy [13], Duren[3] and references therein for basic results on the subject.

Let S_H denote the class of functions $f=h+\overline{g}$ that are harmonic univalent and sense- preserving in the unit disk U for which $f(0)=h(0)=f_z(0)-1=0$. For $f=h+\overline{g}\in S_H$, we may express the analytic functions h and g as

$$h(z)=z+\sum_{k=2}^{\infty}a_k z^k,\ g(z)=\sum_{k=1}^{\infty}b_k z^k,\ |b_1|<1 \qquad (1.1)$$

Observe that S_H reduces to S, the class of all normalized univalent functions, if the co-analytic part of f is zero.

The differential operator $D^n_{\eta,\mu}(\lambda,w)\ (n\in N_0)$ was introduced by Bucur et al.[11]. For $f=h+\overline{g}$, given by (1.1), Sahsene Altinkaya and Sibel Yalcin[6] defined the operator $D^n_{\eta,\mu}(\lambda,w)$ as

$$D^n_{\eta,\mu}(\lambda,w)\ f(z)=D^n_{\eta,\mu}(\lambda,w)\ h(z)+(-1)^n\overline{D^n_{\eta,\mu}(\lambda,w)\ g(z)}\ (1.2),$$

where

$$D^n_{\eta,\mu}(\lambda,w)\ h(z)=z+\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^n a_k z^k$$

and $D^n_{\eta,\mu}(\lambda,w)\ g(z)=\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^n b_k z^k$

,

where $n,\eta\in N_0$, $\mu,\lambda,w\geq 0,\ 0\leq\eta\leq\mu w^{\lambda}$, $D^n_{\eta,\mu}(\lambda,w)\ f(0)=0$.

Recently S. Porwal and Shivam Kumar [12], A.L. Pathak et. al [14] and Rosy et al. [17] have defined and studied a subclass of harmonic univalent functions using differential operator. They have studied the coefficient bounds, Distortion bounds, convolution, convex combination and extreme points.

Motivated by this aforementioned work, in the present paper we define a subclass $B_H(n,\eta,\alpha,\rho,\lambda,w)$ of harmonic univalent functions and study the coefficient estimates, convolution, convex combination and extreme points.

Definition 1.1: We define the subclass $B_H(n,\eta,\alpha,\rho,\lambda,w)$ of functions of the form (1.1) that satisfy the condition

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$$\mathrm{Re}[(1+\rho e^{ir})\frac{D^{n+1}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}-\rho e^{ir}]>\alpha,\ 0\le\alpha<1, r\in R, \rho\ge 0, n,\eta\in N_0 \quad (1.3)$$

where $D^{n}_{\eta,\mu}(\lambda,w)\ f(z)$ is defined by (1.2).

Let $\overline{B_H}(n,\eta,\alpha,\rho,\lambda,w)$ denote that the subclasses of $B_H(n,\eta,\alpha,\rho,\lambda,w)$ which consists of harmonic functions $f_n=h+\overline{g_n}$ such that h and g_n are of the form

$$h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k, g_n(z)=(-1)^n\sum_{k=1}^{\infty}|b_k|z^k. \quad (1.4)$$

II. COEFFICIENT BOUNDS

We begin with a sufficient coefficient condition for functions in $B_H(n,\eta,\alpha,\rho,\lambda,w)$.

Theorem 2.1: Let $f=h+\overline{g}$ is given by (1.1). If

$$\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^n[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k| + \sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^n[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|\le 1-\alpha \quad (1.5)$$

Where, $n,\eta\in N_0$ $\rho\ge 0, 0\le\alpha<1$, $\mu,\lambda,w\ge 0, 0\le\eta\le\mu w^{\lambda}$, ,
then f is sense-preserving, harmonic univalent in U and $f\in B_H(n,\eta,\alpha,\rho,\lambda,w)$.

Proof: If $z_1\ne z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty}b_k z_1^k-\sum_{k=1}^{\infty}b_k z_2^k}{z_1+\sum_{k=2}^{\infty}a_k z_1^k-z_2-\sum_{k=2}^{\infty}a_k z_2^k}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty}b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty}k|b_k|}{1-\sum_{k=2}^{\infty}k|a_k|} \quad (2.2)$$

$$\ge 1-\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^n[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|}{1-\alpha}$$

$$1-\frac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^n[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}$$

≥ 0, which proves univalence.

Note that f is sense-preserving in U.

This is because

$$|h'(z)|\ge 1-\sum_{k=2}^{\infty}k|a_k|z^{k-1}$$

$$>1-\frac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^n[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}$$

$$\ge\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^n[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|}{1-\alpha}$$

$$>\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^n[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k||z|^{k-1}}{1-\alpha}$$

$$\ge\sum_{k=1}^{\infty}k|b_k|z^{k-1}$$

$$\ge|g'(z)|.$$

Using fact that $\mathrm{Re}(w)>\alpha$ if and only if $|1-\alpha+w|\ge|1+\alpha-w|$.

$$w=\frac{(1+\rho e^{ir})D^{n+1}_{\eta,\mu}(\lambda,w)f(z)-\rho e^{ir}D^{n}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}=\frac{A(z)}{B(z)}$$

It suffices to show that

$$|A(z)-(1+\alpha)B(z)|-|A(z)+(1-\alpha)B(z)|\le 0. \quad (2.3)$$

Now find

$$|A(z)-(1+\alpha)B(z)|$$

$$=|(1+\rho e^{ir})D^{n+1}_{\eta,\mu}(\lambda,w)f(z)-\rho e^{ir}D^{n}_{\eta,\mu}(\lambda,w)f(z)-(1+\alpha)D^{n}_{\eta,\mu}(\lambda,w)f(z)|$$

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$$= \left| \begin{array}{l} (1+\rho e^{ir})\left\{ z+\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n+1}a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n+1}b_k(\bar{z})^k \right\} \\ -(1+\alpha+\rho e^{ir})\left\{ z+\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}a_k z^k+(-1)^{n}\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}b_k(\bar{z}^k \right\} \end{array} \right|$$

$$\leq \alpha|z|-\sum_{k=2}^{\infty}\left|[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}\left\{(1+\rho e^{ir})((k-1)(\mu w^{\lambda}-\eta)+k)-1-\rho e^{ir}-\alpha\right\}a_k z^k\right|$$

$$+\sum_{k=1}^{\infty}\left|[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}\left\{(1+\rho e^{ir})((k+1)(\mu w^{\lambda}-\eta)+k)+1+\rho e^{ir}+\alpha\right\}b_k(\bar{z})^k\right|$$

.

and $|A(z)+(1-\alpha)B(z)|$

$$= \left| \begin{array}{l} (1+\rho e^{ir})\left\{ z+\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n+1}a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n+1}b_k(\bar{z})^k \right\} \\ +(1-\alpha-\rho e^{ir})\left\{ z+\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}a_k z^k+(-1)^{n}\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}b_k(\bar{z})^k \right\} \end{array} \right|$$

$$\geq (2-\alpha)|z|-\sum_{k=2}^{\infty}\left|[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}\left\{(-1-\rho e^{ir})((k-1)(\mu w^{\lambda}-\eta)+k)-1+\rho e^{ir}+\alpha\right\}a_k z^k\right|$$

$$-\sum_{k=1}^{\infty}\left|[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}\left\{(1+\rho e^{ir})((k+1)(\mu w^{\lambda}-\eta)+k)-1+\rho e^{ir}+\alpha\right\}b_k(\bar{z})^k\right|$$

.

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$\geq 2(1-\alpha)|z|+2\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)-(\alpha+\rho)]|a_k||z|^k$$

$$-2\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)+(\alpha+\rho)]|b_k||\bar{z}|^k$$

$$> 2(1-\alpha)|z|\left\{ \begin{array}{l} 1-\left\{ \dfrac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k||z|^{k-1}}{1-\alpha} \right. \\ \left. +\dfrac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k||z|^{k-1}}{1-\alpha} \right\} \end{array} \right.$$

This is non-negative by (2.1), and so the proof is complete. The harmonic function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\alpha}{[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]}x_k z^k$$

$$+\sum_{k=1}^{\infty}\frac{1-\alpha}{[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]}\overline{y_k z^k} \quad (2.4)$$

where

$n,\eta \in N_0$, $\rho \geq 0, 0 \leq \alpha < 1$, $\mu, \lambda, w \geq 0, 0 \leq \eta \leq \mu w^{\lambda}$

and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$

Shows that the coefficient bound given by (2.1) is sharp.The functions of the form (2.4) are in $B_H(n,\eta,\alpha,\rho,\lambda,w)$ because

$$\frac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}$$

$$+\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|}{1-\alpha}$$

$$=\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1 \quad .$$

(2.5)

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n=h+\overline{g_n}$, where h and g_n are of the form (1.4).

Theorem 2.2: Let $f_n=h+\bar{g}_n$ be given by (1.4). Then $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$, if and only if

$$\frac{\sum_{k=2}^{\infty}[(k-1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)-(1+\rho)((k-1)(\mu w^{\lambda}-\eta)+k)]|a_k|}{1-\alpha}$$

$$+\frac{\sum_{k=1}^{\infty}[(k+1)(\mu w^{\lambda}-\eta)+k]^{n}[(\alpha+\rho)+(1+\rho)((k+1)(\mu w^{\lambda}-\eta)+k)]|b_k|}{1-\alpha} \leq 1. \quad (2.6)$$

Where $n,\eta \in N_0$, $\rho \geq 0, 0 \leq \alpha < 1$, $\mu, \lambda, w \geq 0, 0 \leq \eta \leq \mu w^{\lambda}$.

Proof: Since $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w) \subset B_H(n,\eta,\alpha,\rho,\lambda,w)$, we only need to prove the 'only if 'part of the theorem. To this end, for functions f_n of the form (1.4), notice that the condition (1.3) is equivalent to

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$$\mathrm{Re}\left[(1+\rho e^{ir})\frac{D^{n+1}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}-\rho e^{ir}\right]>\alpha$$

$$\mathrm{Re}\left[(1+\rho e^{ir})\frac{D^{n+1}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}-(\rho e^{ir}+\alpha)\right]\geq 0$$

$$\mathrm{Re}\left[\frac{(1+\rho e^{ir})D^{n+1}_{\eta,\mu}(\lambda,w)f(z)-(\rho e^{ir}+\alpha)D^{n+1}_{\eta,\mu}(\lambda,w)f(z)}{D^{n}_{\eta,\mu}(\lambda,w)f(z)}\right]\geq 0$$

$$\mathrm{Re}\left[\frac{(1-\alpha)z-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|z^{k}+(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(-1-\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|b_k|\bar{z}^{k}}{z-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k}+(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\bar{z}^{k}}\right]\geq 0$$

$$\mathrm{Re}\left[\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|z^{k-1}-(-1)^{2n}\frac{\bar{z}}{z}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)+\alpha+\rho e^{ir}\right]|b_k|\bar{z}^{k-1}}{1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k-1}+\frac{\bar{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\bar{z}^{k-1}}\right]\geq 0$$

$$\mathrm{Re}\left[\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|z^{k-1}}{1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k-1}+\frac{\bar{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\bar{z}^{k-1}}-\frac{(-1)^{2n}\frac{\bar{z}}{z}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)+\alpha+\rho e^{ir}\right]|b_k|\bar{z}^{k-1}}{1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|z^{k-1}+\frac{\bar{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\bar{z}^{k-1}}\right]\geq 0$$

(2.7)

The above condition (2.7) must hold for all values of z on the positive real axis, where $0\leq|z|=\gamma<1$, we must have

$$\mathrm{Re}\left[\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho e^{ir}\right]|a_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}}-\frac{(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho e^{ir})\left((k+1)(\mu w^{\lambda}-\eta)+k\right)+\alpha+\rho e^{ir}\right]|b_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+(-1)^{2n}\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}}\right]\geq 0$$

Since $\mathrm{Re}\,(e^{ir})=|e^{ir}|=1,$ the above inequality reduces to

$$\left\{\begin{array}{l}(1-\alpha)z-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)-\alpha-\rho\right]|a_k|\gamma^{k-1}\\ -\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}\left[(1+\rho)\left((k+1)(\mu w^{\lambda}+\eta)+k\right)+\alpha+\rho\right]|b_k|\gamma^{k-1}\end{array}\right\}$$

$$\times\left\{1-\sum_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^{n}|a_k|\gamma^{k-1}+\sum_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^{n}|b_k|\gamma^{k-1}\right\}^{-1}\geq 0$$

(2.8)

If the condition (2.6) does not hold, then the numerator in (2.8) is negative for γ sufficiently close to 1. Hence there exist a, $z_0=\gamma_0$ in $(0,1)$ for which the quotient in (2.8) is negative. This contradicts the condition for $f_n\in\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ and so the proof is complete.

III. CONVOLUTION, CONVEX COMBINATION AND EXTREME POINTS

In this section, we show that the class $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ is invariant under convolution and convex combination.
For harmonic functions,

$$f_n(z)=z-\sum_{k=2}^{\infty}|a_k|z^{k}+(-1)^{n}\sum_{k=1}^{\infty}|b_k|\bar{z}^{k}$$

and $F_n(z)=z-\sum_{k=2}^{\infty}|A_k|z^{k}+(-1)^{n}\sum_{k=1}^{\infty}|B_k|\bar{z}^{k}$,

the convolution of f_n and F_n is given by

$$(f_n*F_n)(z)=f_n(z)*F_n(z)=z-\sum_{k=2}^{\infty}|a_kA_k|z^{k}+(-1)^{n}\sum_{k=1}^{\infty}|b_kB_k|\bar{z}^{k}$$

. (3.1)

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Theorem3.1: For $n, \eta \in N_0$, $\rho \geq 0$, $\mu, \lambda, w \geq 0$, $0 \leq \eta \leq \mu w^{\lambda}$, $0 \leq \beta \leq \alpha < 1$, let $f_n \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ and $F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w)$. Then

$$(f_n * F_n) \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w) \subset F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w).$$

Proof : We wish to show that the coefficient of $f_n * F_n$ satisfy the required condition given in theorem (2.2). For $F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\sum_{k=2}^{\infty} \frac{\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right] |a_k| |A_k|}{1-\alpha}$$

$$+ \sum_{k=1}^{\infty} \frac{\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right] |b_k| |B_k|}{1-\alpha}$$

$$\leq \sum_{k=2}^{\infty} \frac{\left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right] |a_k|}{1-\alpha}$$

$$+ \sum_{k=1}^{\infty} \frac{\left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right] |b_k|}{1-\alpha}$$

$$\leq 1.$$

Since $0 \leq \beta \leq \alpha < 1$, and $f_n \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$. Therefore

$$(f_n * F_n) \in \overline{B}_H(n, \eta, \alpha, \rho, \lambda, w) \subset F_n \in \overline{B}_H(n, \eta, \beta, \rho, \lambda, w).$$

We now examine the convex combination of $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$.

Let the function $f_{n_i}(z)$ be defined, for $i = 1, 2, \ldots\ldots, m$, by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} |a_{k,i}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k,i}| \bar{z}^k \quad . \tag{3.2}$$

Theorem3.2 : Let the functions $f_{n_i}(z)$ defined by (3.2) be in the class $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ for every $i = 1, 2, \ldots\ldots, m$. Then the functions $t_i(z)$ defined by

$$t_i(z) = \sum_{i=1}^{\infty} c_i f_{n_i}(z), \quad 0 \leq c_i \leq 1 \tag{3.3}$$

are also in the class $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$, where $\sum_{i=1}^{\infty} c_i = 1$.

Proof : According to the definition of t_i, we can write

$$t_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{k,i}| \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{k,i}| \right) \bar{z}^k.$$

Further, since $f_{n_i}(z)$ are in $\overline{B}_H(n, \eta, \alpha, \rho, \lambda, w)$ for every $i = 1, 2, \ldots\ldots, m$, then

$$\sum_{k=2}^{\infty} \left[(k-1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right] \left(\sum_{i=1}^{\infty} c_i |a_{k,i}| \right)$$

$$+ \sum_{k=1}^{\infty} \left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right] \left(\sum_{i=1}^{\infty} c_i |b_{k,i}| \right)$$

$$= \sum_{i=1}^{\infty} c_i \left[\sum_{k=2}^{\infty} \left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) - (1+\rho)\left((k-1)(\mu w^{\lambda} - \eta) + k\right)\right] |a_{k,i}| \right.$$

$$\left. + \sum_{k=1}^{\infty} \left[(k+1)(\mu w^{\lambda} - \eta) + k\right]^n \left[(\alpha+\rho) + (1+\rho)\left((k+1)(\mu w^{\lambda} - \eta) + k\right)\right] |b_{k,i}| \right.$$

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$$\leq \sum_{i=1}^{\infty} c_i(1-\alpha) \leq (1-\alpha).$$

Next we determine the extreme points of closed convex hulls of $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ denoted by clco $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$.

Theorem3.3 : Let f_n be given by (1.4). Then $f_n \in \overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$,

if and only if $$f_n(z) = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right), \quad (3.4)$$

where

$$h_1(z) = z,$$

$$h_k(z) = z - \left[\frac{(1-\alpha)}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)(k-1)(\mu w^{\lambda}-\eta)+k\right]}\right]z^k$$

$$g_{n_k}(z) = z + (-1)^n\left[\frac{(1-\alpha)}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)(k+1)(\mu w^{\lambda}-\eta)+k\right]}\right]\bar{z}^k$$

, $k = 2,3,.....$ and $k = 1,2,.....$ respectively.

and $\sum_{k=1}^{\infty}(X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0$ .In particular, the extreme points of $\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof : For the functions f_n of the form (3.4) , we have

$$f_n(z) = \sum_{k=1}^{\infty} X_k h_k(z) + Y_k g_{n_k}(z)$$

$$= \sum_{k=1}^{\infty} X_k\left\{z - \left(\frac{1-\alpha}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right)z^k\right\}$$

$$+\sum_{k=1}^{\infty} Y_k\left\{z + (-1)^n\left(\frac{1-\alpha}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right)\bar{z}^k\right\}$$

$$= \sum_{k=1}^{\infty} X_k z + \sum_{k=1}^{\infty} Y_k z$$

$$-\sum_{k=2}^{\infty}\left[\frac{1-\alpha}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right]X_k z^k$$

$$+(-1)^n\sum_{k=1}^{\infty}\left[\frac{1-\alpha}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right]Y_k\bar{z}^k$$

$$= \sum_{k=1}^{\infty}(X_k + Y_k)z - \sum_{k=2}^{\infty}\left[\frac{1-\alpha}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right]X_k z^k$$

$$+(-1)^n\sum_{k=1}^{\infty}\left[\frac{1-\alpha}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right]Y_k\bar{z}^k$$

$$= z - \sum_{k=2}^{\infty}\left[\frac{1-\alpha}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right]X_k z^k$$

$$+(-1)^n\sum_{k=1}^{\infty}\left[\frac{1-\alpha}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right]Y_k\bar{z}^k$$

Then,

$$\sum_{k=2}^{\infty}\frac{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right)\right]^n\left[(\rho+\alpha)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]}{1-\alpha}|a_k|$$

$$+\sum_{k=1}^{\infty}\frac{\left[(k+1)(\mu w^{\lambda}-\eta)+k)\right]^n\left[(\rho+\alpha)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]}{1-\alpha}|b_k|$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$$

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$$= X_1 + \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k - X_1 = \sum_{k=1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k - X_1$$

$= 1 - X_1 \le 1$, and so $f_n \in clco\,\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$.

Conversely, suppose that $f_n \in clco\,\overline{B}_H(n,\eta,\alpha,\rho,\lambda,w)$.

Setting,

$$\text{Setting, } X_k = \frac{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]|a_k|}{1-\alpha},$$

$$0 \le X_k \le 1, k = 2, 3, \ldots\ldots$$

$$Y_k = \frac{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]|b_k|}{1-\alpha},$$

$$0 \le Y_k \le 1, k = 1, 2, \ldots\ldots \tag{3.5}$$

and $X_1 = 1 - \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$ .Therefore f_n can be written as,

$$f_n(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

$$= z - \sum_{k=2}^{\infty} \frac{(1-\alpha)X_k}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]} z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} \frac{(1-\alpha)Y_k}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]} \bar{z}^k$$

$$= z + \sum_{k=2}^{\infty}\left[z + (-z) - \frac{(1-\alpha)z^k}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right] X_k$$

$$+ \sum_{k=1}^{\infty}\left[z + (-z) + \frac{(-1)^n(1-\alpha)\bar{z}^k}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right] Y_k$$

$$= z - \sum_{k=2}^{\infty} zX_k + \sum_{k=2}^{\infty}\left(z - \frac{(1-\alpha)z^k}{\left[(k-1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)-(1+\rho)\left((k-1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right) X_k$$

$$- \sum_{k=1}^{\infty} zY_k + \sum_{k=1}^{\infty}\left(z + (-1)^n \frac{(1-\alpha)\bar{z}^k}{\left[(k+1)(\mu w^{\lambda}-\eta)+k\right]^n\left[(\alpha+\rho)+(1+\rho)\left((k+1)(\mu w^{\lambda}-\eta)+k\right)\right]}\right) Y_k$$

$$= z - z\sum_{k=2}^{\infty} X_k - z\sum_{k=1}^{\infty} Y_k + \sum_{k=2}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_{n_k}(z)Y_k$$

$$= \sum_{k=2}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_{n_k}(z)Y_k + z\left[1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k\right]$$

$$= \sum_{k=2}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_{n_k}(z)Y_k + z\left[1 - \left(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\right)\right]$$

$$= \sum_{k=2}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_{n_k}(z)Y_k + zX_1$$

$$= h_1X_1 + \sum_{k=2}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_{n_k}(z)Y_k$$

$$= \sum_{k=1}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_{n_k}(z)Y_k$$

$$= \sum_{k=1}^{\infty} (h_k(z)X_k + g_{n_k}(z)Y_k), \text{ as required.}$$

(3.6)

This completes the proof the theorem 3.3.

REFERENCES

[1] J. Clunie, T. Sheil – Small, Harmonic Univalent functions, Ann. Acad. Sci. Fenn.Ser. Al. Math., 9, No.3 (1984), 3-25.

[2] O.P. Ahuja, Planer Harmonic Univalent and related mapping, J. Inequal. Pure Appl.Math., 6, No.4 (2005), 122, 1-18, 2005.

[3] P. Duren, Harmonic Mappings in the plane, Cambridge Tract in Mathematics, 156, Cambridge University Press, Cambridge (2004).

[4] B.A. Frasin, Comprehensive family of harmonic univalent functions, SUJ J. Math., 42, No.1 (2006), 145-155.

[5] Y. Avci, E. Zlotkiewicz, On harmonic univalent mappings, Ann. Univ. Mariae Curie SklodowskaSect.A, 44(1990), 1-7.

[6] SahseneAltinkaya and Sibel Yalcin, On a Class of Harmonic Univalent Functions Defined by Using a New Differntial Operator, Th. Appl. of Math. And Comp. Sci. 6 (2) (2016) 125-133.

[7] J.M. Jahangiri, Chan Kim Young, H.M. Srivatsava, Construction of a certain class of harmonic close to convex functions associated with the Alxander integral transform, Integral Transform Spec. Funct., 14, No.3 (2003), 237-242.

[9] Silverman, H. Harmonic Univalent functions with negative coefficients, J. Math. Anal. Appl. 220, 283-289, 1998.

[10] Silverman, H. and Silvia, E.M. Subclasses of harmonic univalent functions, New Zeal. J. Math. 28 (1999), 275-284.

[11] Bucur, R., L. Andrei and D. Daniel (2015).Coefficient bounds and Fekete-Szego problem for a class of analytic functions defined by using differential operator. Appl. Math. Sci. 9, 1355-1368.

[12] S.Porwal and Shivam Kumar, A new subclass of harmonic univalent functions defined by derivative operator, Elect J. Math. Anal. and Appl., (5(1), Jan.2017, 122-134.

[13] Ponnusamy and Rasila, Planar harmonic mappings, RMS Mathematics Newsletter, 17(2) (2007), 40-57, 2007.

[14] A.L. Pathak et. Al, a subclass of harmonic univalent functions associated with the derivative operator, Hacet.J.Mathematics and statistics, 41(1), (2012), 47-58.

[15] J.M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math.Anal. Appl., 235, (1999), 470-477.

[16] E.S. Aqlan, some problems connected with Geometric function theory, Ph.D. thesis (2004), Pune University, Pune.

[17] T. Rosy. B.A. Stephen, K.G. Subramanian and J.M. Jahangiri, Goodman-Ronning-type harmonic univalent functions, Kyungpook Math. J., 41(1) (2001), 45-54.

[18] Sheil-Small, T. Constants for planar harmonic mappings, J. London Math. Soc. 2 (42),237-248, 1990.