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On potentially graphical sequences

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Abstract **- A graphic sequence $\pi=(d_1,\ldots,d_n)$ is potentially $K_4-K_2\cup K_2$-graphic if it has a a realization containing an $K_4-K_2\cup K_2$ as a subgraph where K_4 is a wheel graph on four vertices and $K_2\cup K_2$ is a set of independent edges. In this paper, we find the smallest degree sum such that every n-term graphical sequence contains $K_4-K_2\cup K_2$ as a subgraph**

Key words and Phrases: **Simple graph, potentially graphical sequences**

1 INTRODUCTION

Let $G(V,E)$ be a simple graph (a graph without multiple edges and loops) with n vertices and m edges having vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$. The set of all non-increasing non-negative integer sequences $\pi=(d_1,d_2,\cdots,d_n)$ is denoted by NS_n. A sequence $\pi\in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. There are several famous results, Havel and Hakimi [7, 8] and Erdös and Gallai [2] which give necessary and sufficient conditions for a sequence $\pi=(d_1,d_2,\cdots,d_n)$ to be the degree sequence of a simple graph G. Another characterization of graphical sequences can be seen in Pirzada and Yin Jian Hu [15]. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. We know that a graphic sequence π is potentially K_{k+1}-graphic if and only if π is potentially A_{k+1}-graphic [17]. The disjoint union of the graphs G_1 and G_2 is defined by $G_1\bigcup G_2$. Let K_k and C_k respectively denote a complete graph on k vertices and a cycle on k vertices.

A sequence $\pi=(d_1,d_2,\cdots,d_n)$ is said to be potentially K_{r+1}- graphic if there is a realization G of π containing K_{r+1} as a subgraph. It is shown in [4] that if π is a graphic sequence with a realization G containing H as a subgraph, then there is a realization G of π containing H with the vertices of H having $|V(H)|$ largest degree of π.

In order to prove our main results, the following notations, definitions and results are needed. Let $G=(V(G),E(G))$ be a simple graph with vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$. The degree of v_i is denoted by d_i for $1\le i\le n$. Then $\pi=(d_1,d_2,\cdots,d_n)$ is the degree sequence of G, where $d_1,d_2,\cdots,d_n$ may be not in increasing order. The degree sequence $\pi=(d_1,d_2,\cdots,d_n)$ is said to be potentially A_{r+1}-graphic if it has a realization $H=(V(H),E(H))$, where $V(H)=\{u_1,u_2,\cdots,u_n\}$ and the degree of u_i is d_i for $1\le i\le n$, such that the subgraph induced by $\{u_1,u_2,\cdots,u_{r+1}\}$ is K_{r+1}. In order to prove our main results, we also need the following notations and results. Let $\pi=(d_1,d_2,\cdots,d_n)\in NS_n, 1\le k\le n$. Let

$$\pi^{''}=(d_1-1,\cdots,d_{k-1}-1,\cdots,d_k+1-1,d_k+2,\cdots,d_n),\ if\ \ d_k\ge k,$$

$$=(d_1-1,\cdots,d_k-1,\cdots,d_k+1,\cdots,d_{k-1},d_{k+1},d_n),\ if\ \ d_k<k.$$

Denote $\pi_k^{'}=(d_1^{i'},d_2^{i'},\cdots,d_{n-1}^{i'}),1\le i^{'}\le n$, where

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$d_1^{i'}, d_2^{i'}, \cdots, d_{n-1}^{i'}$ is a rearrangement of the $n-1$ terms of π''. Then π'' is called the residual sequence obtained by laying off d_k from π.

Definition 1.1. A Wheel graph W_n is a graph with n vertices $(n \geq 4)$ formed by connecting a single vertex to all vertices of an $(n-1)$ cycle. A wheel graph on 4 and 5 vertices are shown in Figure 1 below

In 1960 Erdös and Gallai gave the following necessary and sufficient condition.

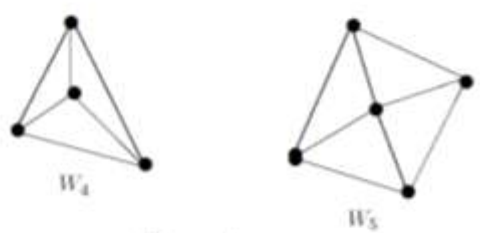

Figure 1

Theorem 1.1.(Erdös, Gallai [2]) Let $n \geq 1$. An even sequence $\pi = (d_1, \ldots, d_n)$ is graphical if and only if

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

is satisfied for each integer k, $1 \leq k \leq n$.

Theorem 1.2. [4] If $\pi = (d_1, d_2, \cdots, d_n)$ is the graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

2 r-GRAPHIC SEQUENCES

The following three results due to Chungphaisan [1] are generalizations from 1-graphs to b-graphs of three well-known results, one by Erdos and Gallai [2], one by Kleitman and Wang [11], one by Fulkerson, Hoffman and McAndrew [5].

Theorem 2.1.([1]) Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of non-negative integers, where the sum of the elements of π is even. Then π is b-graphic if and only if for each positive integer $t \leq n$,

$$\sum_{i=1}^{t} d_i \leq rt(t-1) + \sum_{i=t+1}^{n} \min(rt, d_i).$$

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers with $d_1 \leq \sum_{i=2}^{n} \min(b, d_i)$. Define $\pi_k' = (d_1', \ldots, d_{n-1}')$ to be the nonincreasing rearrangement of the sequence obtained from $(d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n)$ reducing by 1 the remaining largest term that has not already been reduced b times, and repeating the procedure d_k times. π_k' is called the *residual sequence* obtained from π by laying off d_k.

Theorem 2.2.([11]) π is r-graphic if and only if π_k' is r-graphic.

Theorem 2.3. ([1]) Let π be an r-graphic sequence, and let G and G' be realizations of π. Then there is a sequence of r-exchanges, $E_1, \ldots, E_k$ such that the application of these b-exchanges to G in order will result in G'.

An extremal problem for 1-graphic sequences to be potentially K_l^1-graphic was considered by Erdös, Jacobson and Lehel [3], and solved by Gould et al. [6] and Li et al. [14, 13]. Recently, Yin [18] generalized this extremal problem and the Erdös-Jacobson-Lehel conjecture from 1-graphs to b-graphs.

Theorem 2.4. (Yin [19]) Let $n \geq r+s$ and let $\pi = d_1, \ldots, d_n$ be a nonincreasing graphic sequence. If $d_{r+s} \geq r+s-2$, then π is potentially $A_{r,s}$-graphic.

In the same paper Yin published a Havel-Hakimi type algorithm constructing the corresponding $S_{r,s}$-graph.

In 2014 Pirzada and Chat proved the following assertion.

Theorem 2.5.(Pirzada, Chat [16]) If G_1 is a realization of $\pi_1 = d_1^1, \ldots, d_m^1$, containing K_p as a subgraph and G_2 is a realization of $\pi_2 = d_1^2, \ldots, d_n^2$ containing K_q as a subgraph, then the degree sequence $\pi = d_1, \ldots, d_{m+n}$ of the join of G_1 and G_2 is K_{p+q}-graphic.

Problem 2.6. Let H be the graph and n be the

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positive integer. Determine the smallest even integer $\sigma(H,n)$ such that every n-term graphic sequence contains H as a subgraph.

The purpose of this paper is to solve problem 6 by taking $H = W_4 - (K_2 \cup K_2)$ and we also obtain the graphic sequence of the graph when only edge size of the graph and degree of first vertex of the non-increasing sequence of integers is given.

In the following result, we find the smallest graphic sum such that every n-term graphic sequence contains $W_4 - (K_2 \cup K_2)$ as a subgraph.

3 MAIN RESULTS

We Prove the following main result.

Theorem 3.1. If π be the graphic sequence with $\sigma(\pi) \geq 3n-1$ if n is odd and $\sigma(\pi) \geq 3n-2$ if n is even, then π is potentially $W_4 - (K_2 \cup K_2)$-graphic.

Proof. Let π be the graphic sequence. then there exists a graph G which realizes π. We have to show that if $\sigma(\pi) \geq 3n-1$ and $\sigma(\pi) \geq 3n-2$, then every n-term graphic sequence contains $W_4 - (K_2 \cup K_2)$ as a subgraph, where $K_2 \cup K_2$ is the matching in G. To prove the result we use induction on n and we start induction for $n \geq 4$. For $n = 4$, then by the assumption we have $|E| \geq 5$, therefore in this case the realization G of π contains $W_4 - (K_2 \cup K_2)$ as a subgraph as illustrated in figure 1.

Figure 1

Clearly from figure 1, there are exactly two graphs with $|G| = 4$ and $|E| \geq 5$ and both these graphs contains $W_4 - (K_2 \cup K_2)$ as a subgraph. Thus π is potentially $W_4 - (K_2 \cup K_2)$-graphic. Now for $n = 5$, therefore from give assumption we have $|E| \geq 7$, then there are exactly four graphic sequences $(4,3^2,2^2),(4,3^3,1),(3^4,2)$ and $(4^2,2^3)$ with $\sigma(\pi) = 14$ and each of these graphic sequences have a realization G containing $W_4 - (K_2 \cup K_2)$ as a subgraph as illustrated in figure 2.

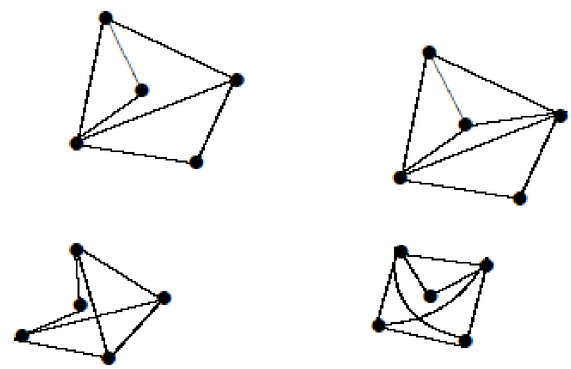

Figure 2

All these graphs contains $W_4 - (K_2 \cup K_2)$ and the result is true in this case also. Now assume that the result is true for all graphic sequences of n-terms and we now consider the graphic sequences of $n+1$ terms. Now if the graphic sequence π contains a vertex of degree equal to 1, then remove it and adjust the new sequence π'. By induction realization G of π' must contain a $W_4 - (K_2 \cup K_2)$. We know that for $n \geq 6$ smallest degree sum such that every n-term graphic sequence contains a clique on three vertices is $2n$. Since $\sigma(\pi) = 16$ for $n = 6$ which is greater than the smallest degree sum such that every 6-term graphic sequence contains a clique on 3 vertices which can be obtained in a realization using the two vertices of highest degree. Let this complete graph graph on three vertices has vertices y_1, y_2 and y_3 and assume that these two vertices of highest degree in the graph are y_1 and y_2. Thus y_1 and y_2 have at least one more adjacency in the graph say y_1 is adjacent to x and y_2 is adjacent to y as shown in figure 4.

Now we consider the following cases

Case I. If $x = y$, then G contains the subgraph

$W_4-(K_2\cup K_2)$, therefore the result is true in this case.

Case II. If $x\neq y$, we consider two subcases.

Subcase 1. Suppose x and y have common vertex w such that xw and $yw\in E(G)$. Then we see that wy_1 and xy_3 are not in realization G of π, since otherwise we get a realization G containing $W_4-(K_2\cup K_2)$. Then by EDT by removing the independent edges $K_2\cup K_2$ (xw and y_1y_3) and inserts the independent edges $K_2\cup K_2$ (wy_1 and xy_3) produces a realization G' of π containing a $W_4-(K_2\cup K_2)$ on the vertex set $S=\{y_1,y_2,w,y\}$.

Subcase 2. Suppose that x and y have no common adjacency of a clique on three vertices. Suppose x is adjacent to x' and y is adjacent to y' such that $x'\neq y'$. Now suppose that $x'y'\notin E(G)$, then by EDT that removes the independent edges $K_2\cup K_2$ (xx' and yy') and inserts the independent edges $x'y'$ and xy produces a realization G of π containing $W_4-(K_2\cup K_2)$. These two subcases are illustrated in Figure 4 and 5 below.

Subcase 3. Now if $x'y'\in E(G)$, then again it is easy to see that the independent edges x_1y_1 and $xy_3\notin E(G)$, since otherwise $W_4-(K_2\cup K_2)$ would exist. Therefore again by EDT that removes the independent edges y_1y_3 and xx' and inserts the independent edges y_1x' and xy_2 produces a realization G of π containing $W_4-(K_2\cup K_2)$. Thus in all cases $W_4-(K_2\cup K_2)$ was produced in some realization of π and therefore the graphic sequence π is potentially $W_4-(K_2\cup K_2)$ and hence the result is proved.

Example 3.2. Let $\pi_1=(4,3^3,1)$ be the non-negative sequence. Then clearly it is graphic with $\sigma(\pi)=14$. Therefore by above theorem realization of π contains $H=W_4-(K_2\cup K_2)$ as a subgraph. Thus π_1 is potentially H-graphic. •

Example 3.3. Let $\pi_2=(3^2,2^3)$ be the non-negative sequence. Then clearly it is graphic with $\sigma(\pi)=12$. Therefore by above theorem every 5-term graphic sequence of π does not contains $H=W_4-(K_2\cup K_2)$ as a subgraph. Thus every 5-term graphic sequence of π_2 does not contain H as a subgraph.

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