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# Some Results on rational functions with Prescribed Poles

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***Abstract*** **- This paper generalize as well as refine some results pertaining to the rational functions with prescribed poles.**
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***Key words and Phrases:*** **Rational function, Polynomial, Poles, Zeros**

## 1 INTRODUCTION

Let $\mathrm{P}_n$ denote the class of all complex polynomials of degree at most $n$. For $a_j \in \mathrm{C}$ with $j = 1,2,...,n$, let

$$W(z) := \prod_{j=1}^{n}(z - a_j)$$

and let

$$B(z) := \prod_{j=1}^{n}(\frac{1-\bar{a}_j z}{z-a_j}), R_n := R_n(a_1, a_2,..., a_n) = \{\frac{P(z)}{W(z)} : P \in \mathrm{P}_n\}.$$

Then $R_n$ is the set of rational functions with poles $a_1, a_2,..., a_n$ at most and with finite limit at $\infty$. We shall always assume that these poles lie in $|z| > 1$.

Note that $B(z) \in R_n$ and $|B(z)| = 1$ for $|z| = 1$. For $r(z) = \frac{P(z)}{W(z)} \in \mathrm{R}_n$, the conjugate transpose $r^*$ of $r$ is defined by $r^*(z) = B(z)\overline{r(\frac{1}{\bar z})}$. The rational function $r \in \mathrm{R}_n$ is called self-inversive if $r^*(z) = \lambda r(z)$ fore some $|\lambda| = 1$.

For a rational function $r \in \mathrm{R}_n$, it is known [[4], Theorem 2] that

$$|r'(z)| + |(r^*(z))'| \leq |B'(z)| \sup_{|z|=1}|r(z)| \quad for\ |z| = 1,$$

Equality holds in (1.1) for $r(z) = \mu B(z)$ with $|\mu| = 1$.

For $r \in \mathrm{R}_n$ to be self-inversive, it is known [[4], Corollary 4] that

$$|r'(z)| \leq \frac{|B'(z)|}{2} \sup_{|z|=1}|r(z)|.$$

More recently, Qasim and Liman [2] considered a more general class of rational functions $r(s(z))$, defined by

$$(r \circ t)(z) = r(t(z)) := \frac{P(t(z))}{W(t(z))},$$

where $t(z)$ is a polynomial of degree $m$ and $r \in \mathrm{R}_n$, so that $r(t(z)) \in \mathrm{R}_{mn}$, and

$$W(t(z)) = \prod_{j=1}^{mn}(z - a_j).$$

Also the Blashke product is given by

$$B(z) = \frac{W^*(t(z))}{W(t(z))} = \frac{\overline{W(t(\frac{1}{\bar z}))}}{W(t(z))} = \prod_{j=1}^{mn}(\frac{1-\bar{a}_j z}{z-a_j}).$$

They proved the following generalizations of (1.1),

***Theorem A.*** If $r(t(z)) \in \mathrm{R}_{mn}$ and all zeros of $t(z)$ lie in $|z| \leq 1$, then for $|z| = 1$,

$$|r^{*'}(t(z))| + |r'(t(z))| \leq \frac{|B'(z)|}{mm'} \sup_{|z|=1}|r(t(z))|,$$

where

$$r^*(t(z)) = B(z)\overline{r(t(\frac{1}{\bar z}))}$$

and

$$m' = \min_{|z|=1}|t(z)|. \qquad (1.1)$$

The result is sharp and equality holds for the rational functions of the form $r(t(z)) = \mu B(z)$ with $|\mu| = 1$ and $t(z) = z^m$.

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In this note, we obtain generalizations of (1.1), (1.2) and (1.3). More precisely we prove

***Theorem 1.*** If $r(t(z)) \in \mathrm{R}_{mn}$ and $|z|=1$, then for every $\beta$ with $|\beta| \leq 1$,

$$|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))|+|B(z)r^{*'}(t(z))t'(z)+\frac{\beta}{2}B'(z)r^{*}(t(z))|$$

$$\leq |B'(z)|\{|1+\frac{\beta}{2}|+|\frac{\beta}{2}|\}\sup_{|z|=1}|r(t(z))|. \qquad (1.4)$$

***Remark 1.*** If we assume $t(z)$ has all its zeros in $|z| \leq 1$ and take $\beta = 0$ in (??) and make use of the Lemma 5, we get (1.3).

For $t(z) = z$, Theorem 1 reduces to the following result.

***Corollary 1.*** If $r \in \mathrm{R}_n$ and $|z|=1$, then for every $\beta$ with $|\beta| \leq 1$,

$$|B(z)r'(z)+\frac{\beta}{2}B'(z)r(z)|+|B(z)(r^{*}(z))'+\frac{\beta}{2}B'(z)r^{*}(z)| \leq |B'(z)|\{|1+\frac{\beta}{2}|+|\frac{\beta}{2}|\}\sup_{|z|=1}|r(z)|. \qquad (1.5)$$

***Remark 2.*** For $\beta = 0$, inequality (1.5) reduces to inequality (1.1).

**Theorem 2.** If $r(t(z)) \in \mathrm{R}_{mn}$ is self-inversive and $|z|=1$, then for every $\beta$ with $|\beta| \leq 1$, we have

$$|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))| \leq \frac{|B'(z)|}{2}\{|1+\frac{\beta}{2}|+|\frac{\beta}{2}|\}\sup_{|z|=1}|r(t(z))|. \qquad (1.6)$$

***Remark 3.*** If we take $\beta = 0$ in inequality (1.6) and make use of the Lemma 5, after supposing that $t(z)$ has all its zeros in $|z| \leq 1$, we get the following result.

**Corollary 2.** If $r(t(z)) \in \mathrm{R}_{mn}$ is self-inversive where $t(z)$ has all its zeros in $|z| \leq 1$, then for $|z|=1$,

$$|r'(t(z))| \leq \frac{|B'(z)|}{2mm'}\sup_{|z|=1}|r(t(z))| \qquad (1.7)$$

where $m' = \min_{|z|=1}|t(z)|$.

**Remark 4.** For $t(z) = z$, inequality (1.7) reduces to inequality (1.2).

## 2 LEMMAS

For the proofs of the theorems we need the following lemmas.

**Lemma 1.** If $r \in R_n$ has $n$ zeros all lie in $|z| \leq 1$, then

$$|r'(z)| \geq \frac{1}{2}|B'(z)||r(z)| \quad for\ |z|=1.$$

The above lemma is due to Li et al. [4].

**Lemma 2.** Let $A$ and $B$ be any two complex numbers then,

$(i)$ if $|A| \geq |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers $\delta$ satisfying $|\delta| < 1$.

$(ii)$ Conversely, if $A \neq \delta B$ for all complex numbers $\delta$ satisfying $|\delta| < 1$ then $|A| \geq |B|$.

The above lemma is due to Li [3].

**Lemma 3.** If $r(t(z)), s(t(z)) \in R_{mn}$ and all the zeros of $s(t(z))$ lie in $|z| \leq 1$ and $|r(t(z))| \leq |s(t(z))|$ for $|z|=1$. Then for every $\beta \in \mathrm{C}$ with $|\beta| \leq 1$ and $|z|=1$, we have

$$|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))| \leq |B(z)s'(t(z))t'(z)+\frac{\beta}{2}B'(z)s(t(z))|. \qquad (2.2)$$

The result is sharp and equality holds in (2.5) for $r(t(z)) = \mu s(t(z))$ with $|\mu| = 1$.

**Proof of Lemma 3.** By Rouche's theorem, the rational function $\lambda r(t(z)) + s(t(z))$ has all its zeros in $|z| \leq 1$ for $|\lambda| < 1$ and has no poles in $|z| \leq 1$. On applying Lemma 1 to $\lambda r(t(z)) + s(t(z))$, we get on $|z|=1$,

$$2|B(z)||\lambda(r(t(z)))'+(s(t(z)))'| \geq |B'(z)||\lambda r(t(z))+s(t(z))|. \qquad (2.3)$$

Now, note that $B'(z) \neq 0$ (e.g, see formula $(14)$ in [4]). So, the right hand side of (2.3) is non zero. Thus, by using $(i)$ of Lemma 2, we have for all $\beta \in \mathrm{C}$ with $|\beta| < 1$,

$$2B(z)(\lambda r'(t(z))t'(z)+s'(t(z))t'(z)) \neq -\beta B'(z)(\lambda r(t(z))+s(t(z))) \quad for\ |z|=1.$$

Equivalently, for $|z|=1$,

$$\lambda(2B(z)r'(t(z))t'(z)+\beta B'(z)r(t(z))) \neq -(2B(z)s'(t(z))t'(z)+\beta B'(z)s(t(z)))$$

$$for\ |z|=1, |\lambda|<1\ and\ |\beta|<1.$$

Using $(ii)$ of Lemma 2, we have

$$|2B(z)r'(t(z))t'(z)+\beta B'(z)r(t(z))| \leq |2B(z)s'(t(z))t'(z)+\beta B'(z)s(t(z))|\quad for\ |z|=1\ and\ |\beta|<1.$$

Now using the continuity for $|\beta|=1$ in (2.4), the desired result follows.

Applying Lemma 3 to the rational function $r(t(z))$ and $B(z)\sup_{|z|=1}|r(t(z))|$, we get the following:

**Lemma 4.** If $r(t(z)) \in R_{mn}$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z|=1$, we have

$$|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))| \leq |B(z)||1+\frac{\beta}{2}|\sup_{|z|=1}|r(t(z))|. \quad (2.5)$$

**Lemma 5.** If $P(z)$ is a polynomial of degree $n$ having all zeros in $\mathbb{T}\cup\mathbb{I}_{-}$, then

$$\min_{z\in\mathbb{T}}|P'(z)| \geq n \min_{z\in\mathbb{T}}|P(z)|. \quad (2.6)$$

The result is best possible and equality in (2.6) holds for polynomials, having all zeros at the origin.
The above lemma is due to Aziz and Dawood [1].

## 3 PROOFS OF THEOREMS

**Proof of Theorem 1.** Let $M := \sup_{|z|=1}|r(t(z))|$. Therefore, for every $\lambda \in \mathbb{C}$ with $|\lambda|>1$, $|r(t(z))| < |\lambda M B(z)|$ for $|z|=1$.
By Rouche's theorem, all the zeros of $G(z)=r(t(z))+\lambda M B(z)$ lie in $|z|<1$. If $H(z)=B(z)\overline{G(\frac{1}{\bar{z}})}$, then $|H(z)|=|G(z)|$ for $|z|=1$ and hence for any $\gamma$ with $|\gamma|<1$, the rational function $\gamma H(z)+G(z)$ has all zeros in $|z|<1$. By applying Lemma 1 to $\gamma H(z)+G(z)$, we have

$$2|B(z)(\gamma H'(z)+G'(z))| \geq |B'(z)||\gamma H(z)+G(z)|,\quad for\ |z|=1. \quad (3.1)$$

Since $B'(z)\neq 0$ therefore the right hand side of (3.1) is non zero. Thus, by using $(i)$ of Lemma 2, we have for all $\beta\in\mathbb{C}$ with $|\beta|<1$,

$$2B(z)(\gamma H'(z)+G'(z)) \neq -\beta B'(z)(\gamma H(z)+G(z)),\quad for\ |z|=1.$$

Equivalently, for $|z|=1$,

$$-\gamma(2B(z)H'(z)+\beta B'(z)H(z)) \neq -(2B(z)G'(z)+\beta B'(z)G(z)),\quad for\ |\gamma|<1, |\beta|<1. \quad (3.2)$$

Using $(ii)$ of Lemma 2 in (3.2), we have

$$|2B(z)G'(z)+\beta B'(z)G(z)| \leq |2B(z)H'(z)+\beta B'(z)H(z)|,\quad for\ |z|=1, |\beta|<1. \quad (3.3)$$

Now by putting $G(z)=r(t(z))+\lambda M B(z)$ and $H(z)=r^{*}(t(z))+\bar{\lambda}M$ in (3.3), we get for $|z|=1$ and $|\beta|<1$,

$$|2B(z)r^{*'}(t(z))t'(z)+\beta B'(z)r^{*}(t(z))+\bar{\lambda}\beta M B'(z)|$$

$$\leq |2B(z)r'(t(z))t'(z)+\beta B'(z)r(t(z))+\lambda B(z)B'(z)(2+\beta)M|. \quad (3.4)$$

By choosing a suitable argument of $\lambda$ and applying Lemma 4 on the right hand side of (??), we get for $|z|=1$ and $|\beta|<1$,

$$|2B(z)r^{*'}(t(z))t'(z)+\beta B'(z)r^{*}(t(z))| - |\lambda||\beta B'(z)|M$$

$$\leq |\lambda||B(z)B'(z)(2+\beta)|M - |2B(z)r'(t(z))t'(z)+\beta B'(z)r(t(z))|. \quad (3.5)$$

Note that $|B(z)|=1$ for $|z|=1$. Making $|\lambda|\to 1$ and using continuity for $|\beta|=1$ in (??), we get the desired result.

**Proof of Theorem 2.** Since $r(t(z))$ is self-inversive, therefore, we have $r^{*}(t(z))=\lambda r(t(z))$ with $|\lambda|=1$. Hence for all $\beta\in\mathbb{C}$,

$$|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))| = |B(z)r^{*'}(t(z))t'(z)+\frac{\beta}{2}B'(z)r^{*}(t(z))|. \quad (3.6)$$

Combining Theorem 1 and (3.6), we have for every $\beta$ with $|\beta|\leq 1$ and $|z|=1$,

$$2|B(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))|$$

$$=|B'(z)r'(t(z))t'(z)+\frac{\beta}{2}B'(z)r(t(z))| + |B(z)r^{*'}(t(z))t'(z)+\frac{\beta}{2}B'(z)r^{*}(t(z))|$$

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$$\leq |B'(z)|\{|1+\frac{\beta}{2}|+|\frac{\beta}{2}|\}\sup_{|z|=1}|r(t(z))|,$$

which proves Theorem 2 completely.

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