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A note on Ultra L-Topologies

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***Abstract*— In this paper, we investigate the lattice structure of the set FX of all Ltopologies on a given finite set X when membership lattice L is a finite Boolean lattice. All the ultra L-topologies and their number in the lattice FX are determined.**

Key words: - L-topology, Lattice, Boolean Lattices, Ultra L-topology.

I. INTRODUCTION

In 1960's many authors like A.K Steiner, Van Rooij studied the lattice structure of the set of all topologies on a given set X. As a result it is known that this lattice is complete, atomic, dually atomic and complemented but neither modular nor distributive in general. Forlich [2] has proved that it is also dually atomic and if $|X| = n$, then there are $n(n-1)$ dual atoms in the lattice of topologies on the set X. Analogously, the lattice structure of the set of L-topologies on a given set came into interest. Johnson [5,6] has investigated lattice structure of the set of L-topologies on a given set X and proved that this lattice is complete, atomic but not modular, not complemented and not dually atomic in general.

In this paper, we investigate the lattice structure of the lattice F_X of all L-topologies on a given finite set X when membership lattice L is a finite boolean lattice. It is easy to see that F_X is complete, atomic but not modular and not distributive. However, in this paper we prove that if $|X| = n$ and $L = 2^m$, then the number of ultra L-topologies in the lattice F_X is $nm(nm-1)$. All the ultra L-topologies are also identified

II. PRELIMINARIES

Throughout this paper, X stands for a finite set having n elements, L for a finite boolean lattice with the least element 0 and the greatest element 1 and F_X stands for the lattice of all L-topologies on X. The constant function in L^X, taking value α is denoted by $\underline{\alpha}$ and x_γ where $\gamma(\neq 0) \in L$, denotes the L- fuzzy point defined by $x_\gamma(y) = \begin{cases} \gamma & if \quad y = x \\ 0 & otherwise \end{cases}$. Any $f \in L^X$ is called as an L-subset of X.

Definition 2.1 An element of L is called an atom if it is a minimal element of $L \setminus \{0\}$.

Definition 2.2 An element of L is called a dual atom if it is a maximal element of $L \setminus \{1\}$.

Definition 2.3 Let (X, F) be an L-topological space and suppose that $g \in L^X$ and $g \notin F$. Then the collection $F(g)$ $\{g_1 \vee (g_2 \wedge g) : g_1, g_2 \in F\}$ is called the simple extension of F determined by g.

Every finite boolean lattice is isomorphic to power set of some set, suppose L is isomorphic to $P(Y)$ where $Y = \{y_1, y_2, y_3, ..., y_m\}$. Then $\alpha_i = \{y_i\}$ and $\beta_i = Y \setminus \{y_i\}$ for $1 \leq i \leq m$ are atoms and dual atoms in L respectively.

Let $\alpha_k = \{y_k\}$ for some $1 \leq k \leq m$ be any atom in L and $A_{\alpha_k} = \{\delta_p = \{y_k, y_p\} : 1 \leq p \leq m$ and $p \neq k\}$ be the set of those $m-1$ elements in L that immediately succeed α_k. Let $\delta_q = \{y_k, y_q\}$ be an arbitrary element of A_{α_k} and L_q^k denotes the sublattice of L generated by the set $\{\alpha_k, \delta_p : \delta_p \in A_{\alpha_k}$ and $p \neq q\}$. Then L_q^k is a

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complete sublattice of L with the least element α_k and the greatest element β_q.

$L \setminus L_q^k$ is also a complete sublattice of L generated by the set $\{\alpha_i, \delta_q : 1 \le i \le m$ and $i \ne k\}$ with the least element 0 and greatest element 1.

Clearly, (i) if $\alpha_k \le \mu$ for some $\mu \in L \setminus L_q^k$, then $\delta_q \le \mu$.

(ii) $\delta_q \wedge \gamma = \alpha_k$, $\forall \gamma \in L_q^k$.

(iii) $L_q^k \cap (L \setminus L_q^k) = \phi$.

Throughout this paper, we will use all these notations.

III. ULTRA L-TOPOLOGY

An L-topology F on X is called an ultra L-topology if the only L-topology on X strictly finer than F is the discrete L-topology.

Remark 3.1 Let U be an L-topology in F_X. In order to show that U is an ultra L-topology, it is sufficient to show that simple extension of U by any L-subset $g \in L^X$ such that $g \notin U$, is the discrete L-topology.

Now certain properties of ultra L-topologies are derived. Let U be an arbitrary ultra L-topology in F_X.

Lemma 3.2 Atleast one L-fuzzy point does not belong to U.

Proof. Suppose all L-fuzzy points belong to U. Since for each $f \in L^X$, $f = \vee x_\lambda$ such that $x_\lambda \le f \Rightarrow f \in U, \forall f \in L^X \Rightarrow U = L^X$, which is a contradiction.

Lemma 3.3 If two L-fuzzy points a_λ and b_η do not belong to U, then $a = b$.

Proof. Suppose the lemma is not true, then the simple extension $U(a_\lambda)$ is an L-topology such that $b_\eta \notin U(a_\lambda) \Rightarrow U(a_\lambda) \ne L^X$. But $U \subset U(a_\lambda)$, which is a contradiction.

Lemma 3.4 There is exactly one element $a \in X$ and one atom $\alpha_i \in L$ for some $1 \le i \le m$ such that $a_{\alpha_i} \notin U$.

Proof. By lemmas 3.2 and 3.3, $\exists$ exactly one element say $a \in X$ such that $a_\lambda \notin U$ for some $\lambda(\ne 0) \in L \Rightarrow a_{\alpha_i} \notin U$ for some $1 \le i \le m$ since L is atomic.

If possible, let $a_{\alpha_i}, a_{\alpha_j} \notin U$ for some $1 \le i, j \le m$ such that $i \ne j$. Then the simple extension $U(a_{\alpha_i})$ is an L-topology such that $a_{\alpha_j} \notin U(a_{\alpha_i}) \Rightarrow U(a_{\alpha_i}) \ne L^X$. But $U \subset U(a_{\alpha_i})$, which is a contradiction.

Theorem 3.5 Let $a \in X$ be an arbitrary element. Then $U_q^k(a) = \{f \in L^X : f(a) \ne \lambda$ for any $\lambda \in L_q^k\}$ is an ultra L-topology such that $a_1 \in U_q^k(a)$ but $a_{\alpha_k} \notin U_q^k(a)$.

Proof. Clearly, $\underline{0}, \underline{1} \in U_q^k(a)$. Let $\{f_i\}_{1 \le i \le r}$ be an arbitrary family of L-subsets in $U_q^k(a)$. Then $f_i(a) \in L \setminus L_q^k$, $\forall 1 \le i \le r$ and since $L \setminus L_q^k$ is a complete sublattice of L

$$\Rightarrow \vee_{i=1}^r f_i(a), \wedge_{i=1}^r f_i(a) \in L \setminus L_q^k \Rightarrow \vee_{i=1}^r f_i, \wedge_{i=1}^r f_i \in U_q^k(a)$$

. Thus, $U_q^k(a)$ is an L-topology.

Clearly, (i) $x_\eta \in U_q^k(a)$, $\forall x (\ne a) \in X$ and $\forall \eta(\ne 0) \in L$.

(ii) $a_\gamma \in U_q^k(a)$, $\forall \gamma(\ne 0) \in L \setminus L_q^k \Rightarrow a_{\alpha_i} \in U_q^k(a)$, $\forall 1 \le i \le m$ such that $i \ne k$.

(iii) $\alpha_k \in L_q^k \Rightarrow a_{\alpha_k} \notin U_q^k(a)$.

Let $g \notin U_q^k(a)$ be an arbitrary L-subset. Then $g(a) = \xi$ for some $\xi \in L_q^k$. Let S = simple extension of $U_q^k(a)$ determined by g. Then $g \in S$

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and $a_1, a_{\delta_q} \in U_q^k(a) \subset S \Rightarrow a_{\alpha_k} \in S \Rightarrow S = L^X$.

Thus simple extension of $U_q^k(a)$ by any of the L-subset not belonging to it, makes a_{α_k} an L-open set. Hence $U_q^k(a)$ is an ultra L-topology.

Remark 3.6 In the theorem 3.5, δ_q can be replaced by any element $\delta_p \in \mathsf{A}_{\alpha_k}$ and corresponding to the element δ_p, the sublattice L_p^k and ultra L- topology $U_p^k(a)$ can be formed in the same way as formed for δ_q. Therefore, theorem 3.5 provides $nm(m-1)$ ultra L-topologies $U_q^k(x)$ where $x \in X$ and $1 \le k, q \le m$ such that $k \ne q$.

Theorem 3.7 Let U be any ultra L- topology in F_X such that $a_1 \in U$ and $a_\lambda \notin U$ for some $a \in X$ and $\lambda(\ne 0,1) \in L$. Then $U = U_q^k(a)$ for some $1 \le k, q \le m$ such that $k \ne q$.

Proof. Case 1 : λ is an atom.

Then $\lambda = \alpha_k$ for some $1 \le k \le m$ and $a_{\alpha_k} \notin U$. By lemma 3.3, $x_\eta \in U$, $\forall x(\ne a) \in X$ and $\forall \eta(\ne 0) \in L$ and by lemma 3.4, $a_{\alpha_i} \in U$, $\forall 1 \le i \le m$ such that $i \ne k$.

Since $a_1 \in U$, there exists no L-subsets $f, g \in U$ such that $f(a) = \alpha_k$ or $f(a) \wedge g(a) = \alpha_k$. Therefore, for atmost one $\delta_q \in \mathsf{A}_{\alpha_k}$, there exists L-subsets $h \in U$ such that $h(a) = \delta_q$ and if $a_{\delta_q} \notin U$, then $U \subset U_q^k(a)$, which is a contradiction. Thus $a_{\delta_q} \in U$.

If there exists some L-subset $f \in L^X$ in U such that $f(a) = \gamma$ for any $\gamma \in L_q^k$, then $a_1, a_{\delta_q}, f \in U \Rightarrow a_{\alpha_k} \in U$, which is a contradiction. Therefore $f \in L^X$ such that $f(a) = \gamma$ for any $\gamma \in L_q^k$, are the only L-subsets not belonging to U.

Hence $U = \{ f \in L^X : f(a) \ne \gamma$ for any $\gamma \in L_q^k \} = U_q^k(a)$, which is an ultra L-topology by theorem 3.5.

Case 2 : λ is not an atom.

Since L is atomic, $a_\lambda \notin U \Rightarrow a_{\alpha_i} \notin U$ for some $1 \le i \le m$ and then by case 1, $U = U_t^i(a)$ for some $1 \le t \le m$ such that $t \ne i$.

Remark 3.8 It is easy to see that if a and b are any two elements of X such that $a \ne b$, then $\mathsf{S}_{a,b_{\alpha_i}} = \{ f \in L^X : f(a) \ne 0 \Rightarrow b_{\alpha_i} \le f \}$ is an L-topology.

Theorem 3.9 Simple extension $\mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_k}) = \{ f \in L^X : f(a) \vee \beta_k = 1 \Rightarrow b_{\alpha_i} \le f \}$ of the L-topology $\mathsf{S}_{a,b_{\alpha_i}}$ by a_{β_k} for some $1 \le k \le m$, is an ultra L-topology.

Proof. Let $f_\lambda = a_\lambda \vee b_{\alpha_i}$, $\forall \lambda(\ne 0) \in L$. Clearly, $f_\lambda \in \mathsf{S}_{a,b_{\alpha_i}} \subset \mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_k})$, $\forall \lambda(\ne 0) \in L$. Then $a_{\beta_k} \wedge f_\lambda = a_\gamma \in \mathsf{S}_{a,b_i}(a_{\beta_k})$, $\forall \gamma(\ne 0) \in L$ such that $\gamma \le \beta_k$. Also $x_\lambda \in \mathsf{S}_{a,b_i}$, $\forall x(\ne a) \in X$ and $\forall \lambda(\ne 0) \in L$. Therefore a_η, where $\eta(\ne 0) \in L$ such that $\eta \vee \beta_k = 1$, are the only L-fuzzy points not belonging to $\mathsf{S}_{a,b_i}(a_{\beta_k})$.

Hence $\mathsf{S}_{a,b_i}(a_{\beta_k}) = \{ f \in L^X : f(a) \vee \beta_k = 1 \Rightarrow b_{\alpha_i} \le f \}$ and $g \in L^X$ such that $g(a) = \ell$ and $g(b) = \mu$, where $\ell, \mu(\ne 0) \in L$ such that $\ell \vee \beta_k = 1$ and $\mu \wedge \alpha_i = 0$, are the only L-subsets not belonging to $\mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_k})$. Simple extension of $\mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_{\beta_k}})$ by any of the L-subsets not belonging to it, makes each a_η, where $\eta(\ne 0) \in L$ such that $\eta \vee \beta_k = 1$, an L-open set. Hence $\mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_{\beta_k}})$ is an ultra L-topology.

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Remark 3.10 Theorem 3.9 provides $nm^2(n-1)$ ultra L-topologies $\mathsf{S}_{x,y_{\alpha_i}}(x_{\beta_k})$ where $x, y \in X$ such that $x \neq y$ and $1 \leq i, k \leq m$.

Theorem 3.11 Let $a \in X$ be an arbitrary element. If U is an ultra L-topology such that $a_1 \notin U$, then $U = \mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_k})$ for some $b(\neq a) \in X$ and $1 \leq i, k \leq m$.

Proof. By lemma 3.3, $x_\lambda \in U$, $\forall x(\neq a) \in X$ and $\forall \lambda(\neq 0) \in L$. $a_1 \notin U \Rightarrow a_{\alpha_k} \notin U$ for some $1 \leq k \leq m$ and by lemma 3.4, $a_{\alpha_i} \in U$, $\forall 1 \leq i \leq m$ such that $i \neq k$ $\Rightarrow a_{\beta_k}, a_\lambda \in U$, $\forall \lambda(\neq 0) \in L$ such that $\lambda \leq \beta_k$.

Clearly, β_k is the only dual atom in L such that $a_{\beta_k} \in U$.

Let $\mathsf{B} = \{\gamma \in L : \alpha_k \leq \gamma\}$. For any $\gamma \in \mathsf{B}$, $\gamma \vee \beta_k = 1$ $\Rightarrow a_\gamma \notin U$. Thus a_γ where $\gamma \in \mathsf{B}$, are the only L-fuzzy points not belonging to U.

Let $\gamma(\neq 1) \in \mathsf{B}$ be an arbitrary element. If possible, let there exists no L-subset in U which assumes value γ at a and let $f \in L^X$ such that $f(a) = \gamma$ and $f(x) = 1$, $\forall x(\neq a) \in X$. Then $f \notin U$ and the simple extension $U(f)$ is an L-topology such that $a_\gamma \notin U(f) \Rightarrow U(f) \neq L^X$. But $U \subset U(f)$, a contradiction. Similar is the case when $\gamma = 1$, in this case consider $f \in L^X$ defined as $f(x) = 0$ for some $x(\neq a) \in X$ and $f(y) = 1$, $\forall y(\neq x) \in X$.

Let $\{f_i\}_{1 \leq i \leq r}$ be the collection of all those L- subsets in U which assumes value γ at a and $g = \bigwedge_{i=1}^{r} f_i$. Then $g \in U$ and since U is an ultra L-topology, $g = a_\gamma \vee b_{\alpha_i}$ for some atom $\alpha_i \in L$ and $b(\neq a) \in X$.

Since $\gamma \in \mathsf{B}$ was an arbitrary element, the same process can be done for any element of B. Let $\delta_1, \delta_2 \in \mathsf{B}$ be two arbitrary elements. Then $\alpha_k \leq \delta_1$ and $\alpha_k \leq \delta_2 \Rightarrow \alpha_k \leq \delta_1 \wedge \delta_2 = \delta_3$ (say). Let $\{h_i\}_{1 \leq i \leq s}$ and $\{g_j\}_{1 \leq j \leq t}$ be the collections of all those L-subsets in U which assume value δ_1 and δ_2 respectively at a. Let $G = \bigwedge_{i=1}^{s} h_i$ and $H = \bigwedge_{j=1}^{t} g_j$. Then $G = a_{\delta_1} \vee b_{\alpha_i}$ and $H = a_{\delta_2} \vee c_{\alpha_j}$, where $b, c(\neq a) \in X$ and $\alpha_i, \alpha_j \in L$. If $b \neq c$ or $i \neq j$, then $G \wedge H = a_{\delta_3} \in U$, a contradiction since $\delta_3 \in \mathsf{B} \Rightarrow b = c$ and $\alpha_i = \alpha_j$.

Therefore $\exists$ a unique element $b(\neq a) \in X$ and a unique atom $\alpha_i \in L$ such that $b_{\alpha_i} \leq h, \forall h \in U$ such that $h(a) = \eta$ where $\eta \in \mathsf{B}$. Clearly, there is no L-subset g_1 in U such that $g_1(a) = \gamma$ and $g_1(b) = \rho$ where $\gamma \in \mathsf{B}$ and $\rho \in L$ such that $\rho \wedge \alpha_i = 0$.

Hence U = $\{f \in L^X : f(a) \vee \beta_k = 1 \Rightarrow b_{\alpha_i} \leq f\} = \mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_k})$, which is an ultra L-topology by theorem 3.10.

Theorem 3.12 Let X be a finite set having n elements and L be a finite boolean lattice isomorphic to the power set $\mathsf{P}(Y)$ where $|Y| = m$, then there are $nm(nm-1)$ ultra L-topologies in the lattice F_X.

Proof. Let U be an arbitrary ultra L-topology in F_X. By lemma 3.2, $a_\lambda \notin U$ for some $a \in X$ and $\lambda(\neq 0) \in L$.

Case 1 : $a_\lambda \notin U$ for some $0 < \lambda < 1$ but $a_1 \in U$. Then by theorem 3.7, $U = U_q^k(a)$ for some $a \in X$ and $1 \leq k, q \leq m$ such that $q \neq k$.

Case 2 : $a_1 \notin U$.

Then by theorem 3.13, U = $\mathsf{S}_{a,b_{\alpha_i}}(a_{\beta_k})$ for some $a, b \in X$ such that $a \neq b$ and $1 \leq i, k \leq m$.

By remarks 3.6 and 3.10, it follows that total number of ultra L-topologies in F_X is $nm(nm-1)$.

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