A Study on Restrained Triple Connected Two Domination Number of a Graph

Abstract: -- The concept of triple connected graphs with real life application was introduced by considering the existence of a path containing any three vertices of a graph $G$. Mahadevan et. al., introduced the concept of triple connected domination number of a graph. In this paper, we introduce a new domination parameter, called restrained triple connected two domination number of a graph. A subset $S$ of $V$ of a non-trivial graph $G$ is said to be a restrained triple connected two dominating set, if $S$ is a restrained two dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all restrained triple connected two dominating sets is called the restrained triple connected two domination number of $G$ and is denoted by $\gamma_{2rtc}(G)$. Any restrained triple connected two dominating set with $\gamma_{2rtc}$ vertices is called a $\gamma_{2rtc}$-set of $G$. We determine this number for some standard and special graphs and obtain bounds for general graph. Its relationship with other graph theoretical parameters is also investigated.

Keywords: -- Triple connected graphs, restrained triple connected, restrained triple connected two domination

I. INTRODUCTION

All graphs considered here are finite, undirected without loops and multiple edges. Unless and otherwise stated, the graph $G = (V, E)$ considered here have $p = |V|$ vertices and $q = |E|$ edges.

A subset $S$ of $V$ of a non trivial graph $G$ is called a dominating set of $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality taken over all dominating sets in $G$. A subset $S$ of $V$ of a non trivial graph $G$ is called a restrained dominating set of $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$ as well as another vertex in $V - S$. The restrained domination number $\gamma_r(G)$ of $G$ is the minimum cardinality taken over all restrained dominating sets in $G$. A subset $S$ of $V$ is said to be two dominating set if every vertex in $V - S$ is adjacent to atleast two vertices in $S$. The minimum cardinality taken over all two dominating sets is called the two domination number and is denoted by $\gamma_2(G)$. A subset $S$ of $V$ is said to be a restrained $2$-dominating set of $G$ if every vertex of $V - S$ is adjacent to at least two vertices in $S$ and every vertex of $V - S$ is adjacent to a vertex in $V - S$. The minimum cardinality taken over all restrained two dominating sets is called the restrained two domination number and is denoted by $\gamma_{2r}(G)$.

A subset $S$ of $V$ of a non trivial graph $G$ is said to be triple connected dominating set, if $S$ is a dominating set and the induced sub graph $<S>$ is triple connected. The minimum cardinality taken over all triple connected dominating sets is called the triple connected domination number and is denoted by $\gamma_{tc}$. A subset $S$ of $V$ of a non trivial graph $G$ is said to be restrained triple connected dominating set, if $S$ is a restrained dominating set and the induced sub graph $<S>$ is triple connected. The minimum cardinality taken over all restrained triple connected dominating sets is called the restrained triple connected domination number and is denoted by $\gamma_{r(tc)}$. A subset $S$ of $V$ of a non trivial graph $G$ is said to be a restrained triple connected two dominating set, if $S$ is a restrained two dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all restrained triple connected two dominating sets is called the restrained triple connected two domination number and is denoted by $\gamma_{2r(tc)}$. Any restrained triple connected two dominating set with $\gamma_{2r(tc)}$ vertices is called a $\gamma_{2r(tc)}$-set of $G$.

Theorem 1.1: A tree is triple connected if and only if $T \cong P_p$, $p \geq 3$.

Theorem 1.2: If the induced subgraph of each connected dominating set of $G$ has more than two pendant vertices, then $G$ does not contain a triple connected dominating set.

Theorem 1.3: For any graph $G$, \[
\left\lceil \frac{p}{\Delta+1} \right\rceil \leq \gamma(G)
\]

II. RESTRAINED TRIPLE CONNECTED TWO DOMINATION NUMBER

Definition 2.1: A subset $S$ of $V$ of a non trivial graph $G$ is said to be a restrained triple connected two dominating set, if $S$ is a restrained two dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all restrained triple connected two dominating sets is called the restrained triple connected two domination number and is denoted by $\gamma_{2r(tc)}$. A subset $S$ of $V$ of a non trivial graph $G$ is said to be restrained triple connected two dominating set, if $S$ is a restrained two dominating set and the induced subgraph $<S>$ is triple connected. The minimum cardinality taken over all restrained triple connected two dominating sets is called the restrained triple connected two domination number and is denoted by $\gamma_{2r(tc)}$.
number of G and is denoted by \( \gamma_{2nc}(G) \). Any restrained triple connected two dominating set with \( \gamma_{2nc} \) vertices is called a \( \gamma_{2nc}^- \) set of G.

**Example 2.2:** For the graph \( G_1 \) in Figure 2.1, \( S = \{v_3, v_4, v_6, v_7\} \) forms a \( \gamma_{2nc}^- \) set. Hence \( \gamma_{2nc}(G) = 4 \).

![Figure 2.1](image1)

**Observation 2.3:** \( \gamma_{2nc}^- \) set does not exists for all graphs if exists \( \gamma_{2nc}(G) \geq 3 \).

**Observation 2.4:** Every \( \gamma_{2nc}^- \) set is a dominating set but not conversely.

**Example 2.5:** For the graph \( G_2 \), in Figure 2.2 \( S = \{v_1\} \) is a dominating set but not a \( \gamma_{2nc}^- \) set.

![Figure 2.2](image2)

**Observation 2.6:** Every \( \gamma_{2nc}^- \) set is a connected dominating set but not conversely.

**Example 2.7:** For the graph \( G_2 \), in Figure 2.2 \( S = \{v_1, v_2\} \) is a connected dominating set but not a \( \gamma_{2nc}^- \) set.

**Observation 2.8:** Every \( \gamma_{2nc}^- \) set is a triple connected dominating set but not conversely.

**Example 2.9:** For the graph \( G_2 \), in Figure 2.2 \( S = \{v_1, v_5, v_6\} \) is a triple connected dominating set but not a \( \gamma_{2nc}^- \) set.

**Observation 2.10:** Every \( \gamma_{2nc}^- \) set is a restrained triple connected dominating set but not conversely.

**Example 2.11:** For the graph \( G_2 \), in Figure 2.2 \( S = \{v_1, v_5, v_6\} \) is a triple connected dominating set but not a \( \gamma_{2nc}^- \) set.

**Observation 2.12:** The complement of the \( \gamma_{2nc}^- \) set need not be a \( \gamma_{2nc}^- \) set.

**Example 2.13:** For the graph \( G_2 \), in Figure 2.2 \( S = \{v_1, v_2, v_5, v_6\} \) is a triple connected dominating set but the complement \( V - S = \{v_3, v_4\} \) is not a \( \gamma_{2nc}^- \) set.

**Theorem 2.14:** For any connected graph G, \( \gamma_r(G) \leq \gamma_c(G) \leq \gamma_{2nc}(G) \).

**2.15 Exact value for some standard graphs:**

- For any path of order \( p \geq 3 \), \( \gamma_{2nc}(K_p) = p \).
- For any cycle of order \( p \geq 3 \), \( \gamma_{2nc}(C_p) = \frac{p}{2} \).
- For any complete graph of order \( p \geq 3 \), \( \gamma_{2nc}(K_p) = p \).
- For any path of order \( p \geq 3 \), \( \gamma_{2nc}(C_p) = \frac{p}{2} \).
- For any cycle of order \( p \geq 3 \), \( \gamma_{2nc}(K_p) = p \).
- For the complete bipartite graph \( K_{m,n} \), \( \gamma_{2nc}(K_{m,n}) = \left\{ \begin{array}{ll} m + n - 2 & \text{if } m \text{ or } n = 2, m \text{ or } n \geq 2 \text{ and } m \text{ or } n \geq 3 \end{array} \right. \)

**Theorem 2.16:** If the induced subgraph of each connected dominating set of G has more than two pendant vertices, then G does not contain a restrained triple connected two dominating set.

The proof follows from Theorem 1.2.

**Theorem 2.17:** For any connected graph G with \( p \geq 3 \) we have \( 3 \leq \gamma_{2nc} \leq p \) and the bounds are sharp.

**Proof:** The lower bound follows from the definition of restrained triple connected two dominating set and the upper bound is obvious. For \( K_p \) the equality of the lower bound is attained and for \( C_p \) the equality of the upper bound is attained.

**Theorem 2.18:** For any connected graph G with five vertices \( \gamma_{2nc}(G) = p - 2 \) if G is isomorphic to any of the following graphs, \( K_5, K_4(1), K_4(2), K_4(3), C_4(3), C_4(4), K_4 - e(3) \).

**Proof:** If G is isomorphic to \( K_5, K_4(1), K_4(2), K_4(3), C_4(3), C_4(4), K_4 - e(3) \) then it can be verified that \( \gamma_{2nc}(G) = p - 2 \). Conversely let G be a connected graph with five vertices and \( \gamma_{2nc}(G) = 3 \). Let \( S = \{v_1, v_2, v_3\} \) be the \( \gamma_{2nc}^- \) set of G. Take \( V - S = \{v_4, v_5\} \) and hence \( \langle V - S \rangle = K_2 \). Also \( \langle S \rangle = P_3 \) or \( C_5 \).

**Case (i) \langle S \rangle = P_3 \text{ and } \langle V - S \rangle = K_2.**

Let \( v_1, v_2, v_3 \) be the vertices of \( P_3 \) and \( v_4, v_5 \) be the vertices of \( K_2 \). Since G is connected \( v_1 \) (or equivalently \( v_3 \)) is adjacent to \( v_4 \) (or \( v_2 \)) is adjacent to \( v_4 \) (or equivalently \( v_3 \)). If \( v_1 \) is adjacent to \( v_4 \) and \( d(v_4) = 3 \) then we can find new graphs by increasing the degrees of \( v_4 \). If \( d(v_4) = 4 \), then \( v_4 \) is adjacent to \( v_1 \) and \( v_2 \) then we can find new graphs by increasing the degrees of \( v_2 \) and we observe that G is isomorphic to \( K_4(1), K_4(2), C_4(3), C_4(4) \). If \( d(v_4) = 4 \), then \( v_4 \) is adjacent to \( v_1, v_2 \), and \( v_3 \). We can find new graphs by increasing the degrees of \( v_5 \) and we observe that G is isomorphic to \( K_4(3), K_4(4), C_4(5) \). If \( d(v_4) = 4 \), then \( v_4 \) is adjacent to \( v_1, v_2 \), and \( v_3 \). We can find new graphs by increasing the degrees of \( v_5 \) and we observe that G is
isomorphic to \( K_4(2), K_4-e(3), K_4(3). \) Suppose \( v_2 \) is adjacent to \( v_4 \) then we can find new graphs by increasing the degrees of \( v_5. \) We observe that \( G \) is isomorphic to \( K_4(1), C_4 \) (3), \( K_4(2), K_4(1) \) and \( K_4(2), K_4-e(3), \) and \( K_4(3). \)

**Case (ii)** \( <S> = C_3 \) and \( <V-S> = K_2.\)

Let \( v_1, v_2, v_3 \) be the vertices of \( C_3 \) and \( v_4, v_5 \) be the vertices of \( K_2. \) Since \( G \) is connected \( v_4 \) or \( v_5 \) is adjacent to \( C_3. \) If \( d(v_4) = 3, \) then \( v_4 \) is adjacent to \( (v_1 \) and \( v_2) \) or \( (v_1 \) and \( v_3) \) or \( (v_2 \) and \( v_3) \). If \( v_4 \) is adjacent to \( v_1 \) and \( v_2 \) then we can find new graphs by increasing the degrees of \( v_5. \) We observe that \( G \) is isomorphic to \( K_4(2), C_4(4) \) or \( K_4-e(3) \) and \( K_4(3), K_4-e(3), K_4(2) \) and \( K_4(3). \) If \( d(v_4) = 4, \) then \( v_4 \) is adjacent to \( v_1, v_2 \) and \( v_3. \) By increasing the degree of \( v_5 \) \( G \) is isomorphic to \( K_3(1) \) and \( K_5. \)

**Theorem 2.19:** Let \( G \) be a graph such that \( G \) and \( \overline{G} \) have no isolates of order \( p \geq 3. \) Then

- i. \( \gamma_{2mc}(G) + \chi_{2mc}(G) \leq 2p \)
- ii. \( \gamma_{2mc}(G) + \chi_{2mc}(G) \leq p^2 \) and the bound is sharp.

**Proof:** The bound directly follows from Theorem 2.17. For cycle \( C_p \) and path \( P_p \) equality of both the bounds are attained.

**Relation with other graph parameters:**

**Theorem 2.20:** For any connected graph \( G, \gamma_{2mc}(G) + \chi(G) \leq 2p \) and the inequality holds if and only if \( G \) is isomorphic to \( K_3 \) or \( K_4. \)

**Proof:** It is clear that \( \gamma_{2mc}(G) \leq p \) and \( \chi(G) \leq p. \) Thus, \( \gamma_{2mc}(G) + \chi(G) \leq p + p = 2p. \) Suppose \( G \) is isomorphic to \( K_3 \) or \( K_4. \) Then clearly \( \gamma_{2mc}(G) + \chi(G) = 2p. \) Conversely, let \( \gamma_{2mc}(G) = 2p. \) The only possible case is \( \gamma_{2mc}(G) = p \) and \( \chi(G) = p. \) If \( \chi(G) = p \) then \( G \) is isomorphic to \( K_p. \) In \( K_p, \gamma_{2mc}(G) = 3, p \neq 4 \) and \( \gamma_{2mc}(K_4) = 4, \) so that \( G \) is isomorphic to \( K_3 \) or \( K_4. \) Also if \( G \) is isomorphic to \( C_3, \) then \( \gamma_{2mc}(G) = p \) and \( \chi(G) = p. \) is possible.

**Theorem 2.21:** For any connected graph \( G, \gamma_{2mc}(G) + \chi(G) \leq 2p - 1 \) and the equality holds if and only if \( G \) is isomorphic to \( K_3 \) or \( K_4. \)

**Proof:** It is clear that \( \gamma_{2mc}(G) \leq p \) and \( \chi(G) \leq p - 1. \) Thus, \( \gamma_{2mc}(G) + \chi(G) \leq p + p - 1 = 2p - 1. \) Suppose \( G \) is isomorphic to \( K_3 \) or \( K_4. \) Then clearly \( \gamma_{2mc}(G) + \chi(G) = 2p - 1. \) Conversely, let \( \gamma_{2mc}(G) + \chi(G) = 2p - 1. \) The only possible case is \( \gamma_{2mc}(G) = p \) and \( \chi(G) = p - 1. \) Since \( \chi(G) = p - 1, \) \( G \) is a complete graph. In \( K_p, \gamma_{2mc}(G) = 3, p \neq 4 \) and \( \gamma_{2mc}(K_4) = 4. \) Hence \( G \) is isomorphic to \( K_3 \) or \( K_4. \)

**Theorem 2.22:** For any connected graph \( G \) with \( p \geq 3 \) vertices, \( \gamma_{2mc}(G) + \Delta(G) \leq 2p - 1 \) and the bound is sharp.

**Proof:** Let \( G \) be a connected graph with \( p \geq 3. \) vertices. We know that, \( \Delta(G) \leq p - 1 \) and by Theorem 2.17 \( \gamma_{2mc}(G) \leq p. \) Hence \( \gamma_{2mc}(G) + \Delta(G) \leq 2p - 1. \) For \( K_3 \) and \( K_4 \) the bound is sharp.