IFERP

ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 4, Issue 6, June 2019

Double Sequences & Series

[1] Linty Antony T, [2] Nice C A, [3] Selina V P
[1] AP-IES College of Engineering, [2] AP-IES College of Engineering, [3] AP-IES College of Engineering
[1] lintyies@gmail.com, [2] niceabhrahamc@mailcom, [3] vpselin1984@gmail.com

***Abstract:* - In this study we define the double sequences & series. In mathematics a sequence is an enumerated collection of objects in which repetitions are allowed. Like a set, it contains elements. The number of elements is called length and same elements can appear multiple times. Formally, a sequence can be defined as a function whose domain is a set of all natural numbers. The concept of Double sequences & series us an extension of single sequence & series. First we discuss about the single sequence and then extend them to the concept of Double sequences & series. Then discuss about the some results and examples related to them.**

1. INTRODUCTION

George cantor, the creator of the set theory ,made a considerable contribution to the development of the theory of real sequences.

A sequence of real numbers is a function defined on a set of natural numbers, whose range is contained in the set of real numbers.

If X **:** N→R is a sequence, the image is usually denoted by $\boldsymbol{x_n}$ and is known as the n^th^ term of the sequence.

A series is an expression that can be written in the form $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \dots\dots + u_n + \cdots$

The numbers $u_1, u_2, u_3, \dots$ ..are called the terms of the series. The sum $s_n = u_1 + u_2 + \dots\dots + u_n$ is called the n^th^ partial sum of the series $\sum_{n=1}^{\infty} u_n$. If s_n is the n^th^ partial sum of the series $\sum_{n=1}^{\infty} u_n$,then the sequence is called the sequence of partial sums of the series.

Let { s_n } be the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \dots\dots + u_n + \cdots$

If the sequence { s_n } converges to a limit S, then the series is said to converges to S and S is called the sum of the series. We denote this by writing S = $\sum_{n=1}^{\infty} u_n$.If the sequence of partial sums diverges then the series is said to diverges.

The theory of Double sequences & series is an extension of single sequence & series. The theory of

Double series is related to the theory of double sequence. To each double sequence there are three important sums

- $\sum_{n,m=1}^{\infty} Y(n,m)$
- $\sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} Y(n,m))$
- $\sum_{m=1}^{\infty} (\sum_{n=1}^{\infty} Y(n,m))$

The following points are discuss in this paper,

- ❖ Double sequence & examples
- ❖ Convergence of double sequences
- ❖ Double series
- ❖ Convergence Methods for double series
- ❖ Applications

2. DOUBLE SEQUENCE

In this section we define double sequence of real or complex numbers.

Double sequence of real or complex numbers is a function defined on N x N whose range is set of real or complex numbers.

If X : N x N →C, the image X(n, m) of a positive integer n, m is usually denoted by $x_{n,m}$.For a fixed n, (x_{n1}, x_{n2}, x_{n3} ….x_{nm}……..) forms the n^th^ row of the double sequence. For a fixed m, ($x_{1,m}, x_{2,m}, x_{3,m}, \dots x_{n,m}$……..) forms the m^th^ column of the double sequence.

➢ Convergence of Double Sequence

We say that the double sequence X(n, m) converges to a limit ' l ', and we write $\lim_{n,m\to\infty} x_{n,m} = l$ if the following conditions are satisfied:

" For every ε > 0, ϶ N = N(ε) such that | $x_{n,m}$m- a| < ε for every n or m ≥ N". The number *l* is known as the double limit of the double sequence. we can say that the sequence X(n, m) diverges, if no such limit exist. Next we define the Row limit & Column limit of a double sequence.

- Row limit : If $\lim_{n\to\infty}(\lim_{m\to\infty} x_{n,m})$ exist it is called the row limit of the double sequence.
  Notation is $\lim_{n\to\infty} R(n)$
- Column limit : If $\lim_{n\to\infty}(\lim_{m\to\infty} x_{n,m})$ exist it is called the column limit of the double sequence.

Notation is $\lim_{m\to\infty} C(n)$

SOME IMPORTANT RESULTS

- $\lim_{n,m\to\infty} X_{n,m}$ = β. For each fixed m, let the limit$\lim_{n\to\infty} x_{n,m}$) exists. Then the iterated limit$\lim_{m\to\infty} x_{n,m}$) also exist and has the value β.
- A double sequence of complex numbers can have at most one limit.

International Journal of Science, Engineering and Management (IJSEM)
Vol 3, Issue 6, June 2019

- A double sequence X(n, m) is said to be bounded ,if there exist a real number M >0, such that | X(n,m)| ≤ M , for all n,m ϵ N.

3. DOUBLE SERIES

In this section we introduce the new concept Double series and we shall give the conditions for their convergence. Also discuss the some example for series convergence tests.

Definition: Let X: N x N →C be a sequence of complex numbers and let X(n,m) be the double sequence represented by the equation

X(n,m) = $\sum_{i=1}^{n}\sum_{j=1}^{m} a(i,j)$. The pair (a,x) is called the *double series* and is represented by $\sum_{n,m=1}^{\infty} a_{n,m}$.

Each number a(n,m) is called the *term* of the series and X(n,m) is known as partial sum.

If $\lim_{n,m\to\infty} X_{n,m} = l$, we can say that the double series $\sum_{n,m=1}^{\infty} a(n,m)$ is *convergent* . If no such limit exist the double series is *divergent*.

The series $\sum_{n=1}^{\infty}[\sum_{m=1}^{\infty} Z(n,m)]$ and $\sum_{m=1}^{\infty}[\sum_{n=1}^{\infty} Z(n,m)]$ are known as Iterated series.

Let the general row sum be $R_m = \sum_{n=1}^{\infty} a_{n,m}$ for m = 1,2,3,……Then R = $\sum_{m=1}^{\infty} R_m$ is called the sum by rows. In a similar manner , Let $C_n = \sum_{m=1}^{\infty} a_{n,m}$ for n = 1,2,3,…….Then C = $\sum_{n=1}^{\infty} C_n$ is called the sum by column of the double series.

Example . 1.

The double series $\sum_{n,m=1}^{\infty}\frac{1}{2^n3^m}$ is convergent. For each n,m ϵ N the partial sum

X(n,m) $\leq\left[\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right]\left[\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^m}\right]$ since $\sum\frac{1}{2^n}$ *and* $\sum\frac{1}{3^m}$ are convergent. $\exists M>0$ such that $\left[\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right]\left[\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^m}\right]\leq 0$ for every n,m ϵ N. It follows that 0 ≤ X(n,m) ≤ M. Thus the set $\{X(n,m): n,m\epsilon$ N} is bounded and hence the given series is converges.

Notes:

- If the double series $\sum_{n,m=1}^{\infty} a(n,m)$ is convergent then $\lim_{n,m\to\infty} a(n,m) = 0$
- A double series $\sum_{n,m=1}^{\infty} a(n,m)$ of complex numbers converges if its sequence of partial sums X(n,m) is Cauchy. It is known as Cauchy convergence criterion for double series.
- When the double series $\sum_{n,m=1}^{\infty} a(n,m)$ converges then $a_{n,m}\to 0$ *as* n,m → ∞,but converse is not true.

For example: Let the double series be $\sum_{n,m=1}^{\infty}\frac{1}{m}+\frac{1}{n}$.

Let $\lim_{n,m\to\infty} a_{n,m} = \lim_{n,m\to\infty}\frac{1}{m}+\frac{1}{n} = 0$

Now we show that converse is not true .$X_{n,m} = (1+1)\left(\frac{1}{2}+\frac{1}{2}\right)+\cdots\left(\frac{1}{m}+\frac{1}{n}\right)$

$= 1+\frac{1}{2}+\ldots\ldots+\frac{1}{m}+ 1+\frac{1}{2}+\ldots\ldots+\frac{1}{n} >$ m$\frac{1}{m}+\frac{1}{n} = 2$

So that $\lim_{n,m\to\infty} X_{n,m}$ does not exist.

Hence the above series is divergent.

CONVERGENCE TESTS:

We shall provide the different types of convergence tests for double series. These yield convergence tests for double series that are analogous to well-known convergence tests for single series.

1 Comparison Test
2 Limit Comparison Test
3 Ratio Test
4 Abel's (m,n)th term test

1. Comparison Test

Let $\sum_{n,m=1}^{\infty} a(n,m)$ *and* $\sum_{n,m=1}^{\infty} b(n,m)$ be double series with non negative terms and $a(n,m) \leq b(n,m)$ then,

(a). if the bigger series $\sum_{n,m=1}^{\infty} b(n,m)$ converges then the smaller series $\sum_{n,m=1}^{\infty} a(n,m)$ also converges.

(b). if the smaller series $\sum_{n,m=1}^{\infty} a(n,m)$ diverges then the bigger series $\sum_{n,m=1}^{\infty} b(n,m)$ also diverges.

Example:

** Consider the double series $\sum_{n,m=1}^{\infty}\sin(\frac{1}{2^n3^m})$ is convergent ,Indeed we have $\sin\left(\frac{1}{2^n3^m}\right)\leq\frac{1}{2^n3^m},\ for\ all\ n,m\epsilon$ N.But we know that $\sum\frac{1}{2^n3^m}$ is convergent. Then by comparison test the given series convergent.

** Consider the double series $\sum_{n,m=1}^{\infty}\cos(\frac{1}{2^m3^n})$ is convergent , Indeed we have $\cos\left(\frac{1}{2^m3^n}\right)\leq\frac{1}{2^m3^n},\ for\ all\ n,m\epsilon$ N.But we know that $\sum\frac{1}{2^m3^n}$ is convergent. Then by comparison test the given series convergent.

2. Limit Comparison Test

Let $(a_{n,m})$ and $(b_{n,m})$ be double series With non negative terms for all (n,m) ϵ N^2 each row series and each column series corresponding to both $\sum_{n,m=1}^{\infty} a(n,m)$ *and* $\sum_{n,m=1}^{\infty} b(n,m)$ are convergent and $\lim_{n,m\to\infty}\frac{a_{m.n}}{b_{m,n}}$ = p where p ϵ R and p≠ 0.Then

a) $\sum_{n,m=1}^{\infty} a(n,m)$ is convergent if and only if $\sum_{n,m=1}^{\infty} b(n,m)$ convergent.

b) $\sum_{n,m=1}^{\infty} a(n,m)$ is divergent if and only if $\sum_{n,m=1}^{\infty} b(n,m)$ divergent

3. Ratio Test

Let $a(n,m)$ be a sequence of non zero numbers such that either

$\frac{a_{n,m+1}}{a_{n,m}} \to p$ *or* $\frac{a_{n+1,m}}{a_{n,m}} \to$ q as n,m $\to \infty$

where p,q ϵ R U (∞)

Suppose that each row series and column series corresponding to $\sum_{n,m=1}^{\infty} a(n,m)$ are

1) convergent if P < 1 or q < 1 .
2) divergent if P > 1 or q > 1 then
3) P = 1 or q = 1, test fails.

Example: Consider the series $\sum \frac{1}{2^n 3^m}$

Let $a_{n,m} = \frac{1}{2^n 3^m}$ and $a_{n+1,m} = \frac{1}{2^{n+1} 3^m}$

$\frac{a_{n+1,m}}{a_{n,m}} = \frac{2^{n+1} 3^m}{2^n 3^m} = \frac{1}{2} < 1$

Hence by Ratio test, the given series is converges

OR

Let $a_{n,m} = \frac{1}{2^n 3^m}$ and $a_{n,m+1} = \frac{1}{2^n 3^{m+1}}$

$\frac{a_{n,m+1}}{a_{n,m}} = \frac{2^n 3^{m+1}}{2^n 3^m} = \frac{1}{3} < 1$

Hence by Ratio test, the given series is converges

4. Abel's (m,n)th term test

Suppose that $(a_{n,m})$ is monotonically decreasing double sequence of non negative terms .If the double series $\sum_{n,m=1}^{\infty} a(n,m)$ is convergent, then mn $a_{n,m} \to 0$ as $n, m \to \infty$

This (m,n)th term test is useful for establishing the divergence of a double series.

REFERENCES

1. M. Mursaleen & S.A Mohuiddinne "Convergence Methods for double sequences and Applications
2. Rudin. W," Principles of Mathematical Analysis, third edition Mc Graw-Hill,1976
3. www/2.iugaza.edu.ps/…volume%2014-%20Issue%201%
4. Robinson G.M..Divergent double sequences and series.Trans.Amer.Math.Soc.28