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Renewal Process & Markov Renewal Process

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Abstract: **- A Study on Stochastic modal is a tool for estimating probability distributions of potential customers by allowing for random variation in one or more inputs over time. Stochastic modal provides an approximation to the real world situation. A modal involving a random variable or chance factor is called Stochastic modal. Here we discuss mainly the renewal process and the Markov renewal process . It includes the basic definitions of renewal proces , types of renewal process and their examples. Markov renewal process is considered as a generalizations of all other process.**

Index Terms— Markov property ,Random variable , Renewal theory, stochastic process.

1. INTRODUCTION

Consider the following examples, In an interval of (0,t], N(t) denotes the no: of (1) Customers arriving at a store (2) Accidents occurring in a city being reported to the central police station (3) Light bulbs being replaced as soon as they burn out (4) A machine component being replaced as soon as it fails etc.

N(t) is a non-decreasing discrete state process that counts the no: of events occurring in (0,t] and hence is called a Counting process.

Let the time interval between concecutive epochs of occurrence of the event be independent identically distributed random variable. Then N(t) is called a Renewal Counting Process and we call the event of interest a Renewal Event . The process composed of the inter occurrence of renewal events is called the Renewal Process.

The study of the renewal process is directed towards gaining information on the distribution of the time between renewal events and the following major characteristics:

(i) Distribution of N(t)

(ii) Expected no: of renewals at time t, E(N(t)) which is known as the renewal function

(iii)The probability mass or the probability density function related to a renewal at a given time point.

(iv) Time needed for a specific no: of events to occur.

(v)Distribution of time since the last epoch of occurrence and time until the next epoch of occurrence of the renewal event

As other stochastic process ,the time parameter can be considered to be either discrete or continuous.

2. DISCRETE RENEWAL PROCESS

Con Consider a sequence of independent and repeated trials with possible outcomes E_i , i=1,2,3,… Suppose a certain outcome E* is of particular interest. With regard to this outcomes ,we say that renewal occurs at the n^{th} trial iff E* occurs at the n^{th} trial . The time interval between any two consecutive epochs of occurrence of E* is known as the Renewal Period for the Process.

Let be $p^{(n)}$ the probability that E * occurs at the n^{th} trial .Assume that $p^{(0)} = 1$. Let $f^{(n)}$ be the probability that E* occurs for the first time at n^{th} trial n≥1. We may take $f^{(0)} = 0$ and further let

$f^{*}= \sum_{n=1}^{\infty} f^{(n)}$. Clearly f^{*} is the probability of eventual occurrence of E^{*} .

Example

A simple example of a renewal counting process when time is discrete is the Bernoulli Process.

Let ‘a’ be the probability that of success at any Bernoulli trial and ‘b’ be the probability of failure so that a + b =1.

Suppose that we are interested in the occurrence of success which is denoted by the event E^{*}. Since the trial are independent ,the probability that E^{*} occurs at any trial is ‘a’ and independent of ‘n’ . The distribution $f^{(n)}$ of the renewal period is given by the Geometric distribution.

$f^{(n)} = ab^{(n-1)}$,n=1,2,…

Clearly f^{*}=1

The probability distribution of the no: of renewal in n-trials is Binomial with probability of success ‘a’ and the Expected no: of renewal in n trial is na.

Discrete Renewal Equation

We have previously assumed that $p^{(0)} = 1$

That is the initial time point is the epoch at which E^{*}occurs. This may not be a realistic assumption in some situation.

To remove this we consider the initial renewal period as having a distribution $b^{(n)}(k)$,

$b^{(n)}(k)$ = P{ E^{*}occurs for the first time at trial n / process initiated at k trial before time period 0}n=1,2,… .

Essentially we think of k being fixed and the waiting time up to the first occurrence of E*having a distribution given by the probability $\{b^{(n)}\}$. By introducing $\{b^{(n)}\}$ is to allow the distribution of the initial renewal period to differ from the distribution of subsequent renewal period . These subsequent renewal period have distribution given by $\{f^{(n)}\}$.

The probability $\{b^{(n)}\}$,$\{p^{(n)}\}$,$\{f^{(n)}\}$ can all be related by considering the trial no: r (r=0,1,2,3,…n) at which the previous renewal occurred .

For thus we write $p^{(n)}=P(\cup_{r=0}^{n} A_r)$, n≥1

Where $A_r=$ $\{E^*$ *occured previously at trial no*$:(n-r)$ *and the next time at trial* $n\}$

The events $\{A_r\}$ are mutually exclusive and

P(A_r) =$p^{(n-r)}f^{(r)}$ for r=1,2,3,… The case r=0 corresponds to E^* occurring for the first time at trial n, with the probability $b^{(n)}$.

$$p^{(n)} = b^{(n)} + p^{(n-1)}f^{(1)} + \cdots + p^{(1)}f^{(n-1)} + p^{(0)}f^{(n)}$$

This equation is known as Discrete Renewal Equation.

3. CONTINUOUS RENEWAL PROCESS

Let an event E^* occur at t_1, t_2,....and $z_r = t_r - t_{r-1}$, r=1,2,3,…..be independent and identically distributed random variables with

$$P(\leq x) = F(x), E(z_r = \mu)$$

And further let $t_0 = 0$ and $z_1 = t_1 - t_0$be distributed as $P(z_1 \leq x) = F_1(x)$

Let N(t) be the no: of times E^* occur in time (0,t] . The process N(t) is a renewal counting process in continuous time and z_r are the renewal periods.

4. MODIFIED RENEWAL PROCESS & ALTERNATING RENEWAL PROCESS

Consider a sequence $\{X_n\}$ of independent random variable where X_0 has the distribution F_0(x) but X_i (i≠0) has distribution F_i defined above. Here X_0 may be thought of as the life time of the article in use at time t=0 where distribution is in general different from F.

We describe the state of a computer going through two periods alternately work and repair. We assume that those periods could be modeled as having Exponential Distributions . Suppose that the distribution needed for the modal are not exponential .Then the resulting process is not Markovian. If we retain the assumptions of the independence between neigh-bouring periods we can use the renewal process modal to analyze the system. Because of the alternating nature of the renewal periods , we call the resulting process an alternating renewal process.

In an alternating renewal process let the alternating periods be denoted by X's and Y's respectively. $(X_1, X_2, ...)$ is the sequence of work periods (busy periods) and $(Y_1, Y_2, ...)$ is a sequence of repair periods (idle periods) respectively. If consider $Z_i = X_i + Y_i$ = Busy period + idle period which form a busy cycle . Thus $(Z_1, Z_2, ...)$ form a sequence of iid random variables constitutes a regular renewal process which alternates between X –states and Y –states .

Let $U^*{}_1(\theta)$ be the renewal function for the renewal process $\{Z_1\}$ when the renewal is marked by a transition from a Y-state to an X-state. Similarly let $U^*{}_2(\theta)$ be the renewal function for the renewal process when the renewal is marked by a transition from an X- state to a Y-state.

Example

In a Queueing system Busy period during which service is rendered alternate with idle periods during which the server is idle . A common practice is to start serving as soon as a customer arrives at an empty system and not to stop serving as long as there are customers in the system. With this structure , time periods between successive epochs when busy periods starts are indent identically distributed random variables thus defining a renewal process . If in addition we assume that there is only one server and the arrivals are in a Poisson Process , The idle period as an Exponential Distribution Which is independent of the preceding Busy period. No we have an alternating renewal process with the Busy and Idle periods and the component renewal periods.

5. RENEWAL REWARD PROCESS

In applied problems the process of interest is often not the renewal process itself, but an associated process that is triggered by renewal events .In a service system with every completed service one ca associate a profit (cost) and this secondary process is crucial in the operation of the system. In this situation we define a cumulative process also known as Renewal Reward process.

Let $(Z_1, Z_2, ...)$ define the renewal process as before. Let $t_0, t_1,$ …..be the renewal epochs and N(t) be the renewal counting process. Associate process composed of i id Random Variables Y_i, i=1,2,… at renewal epochs t_i, i=1,2,.. respectively .

Let $$Y(t) = \sum_{i=1}^{N(t)+1} Y_i = \sum_{n=0}^{N(t)} Y_n$$ = total reward up to time t

The Y(t) process is called a renewal reward process or compound renewal process.

For example if N(t) denotes the no: of floods up to time t and Y_i denotes the damage due to the i[th] flood then Y(t) denotes the total damage due to flood up to time t.

By the definition of Y(t) , if E(Y) and E(X) are finite as,

$\frac{Y(t)}{t} = \sum_{n=0}^{N(t)} Y_n /$ N(t) x N(t)/t $\rightarrow E(Y)$ x 1/E(X) asymptotically . Since by the strong law of large no: the firs $t \rightarrow \infty$ t term $\rightarrow E(Y)$ and by the property as $\rightarrow \infty$,

N(t)/t $\rightarrow$ asymptotically

$$f(x) = \begin{cases} \frac{1}{\mu}, & if 0 < \mu < \infty \\ 0, & if\ \mu = \infty \end{cases}$$

the second term $\rightarrow E(X)^{-1}$

6. MARKOV RENEWAL PROCESS

Consider a system which move from one state to another with random sojourn time in between , the successive states visited from a Markov Chain and a Sojourn has a distribution which depends on the state being visited as well as the next state to be entered . such process are termed as Markov Renewal process.

Definition

A Stochastic process (X,T) = $\{X_n, T_n : n \in N\}$ is said to be a Markov Renewal process with state space E provided that

P$\{X_{n+1} = j, T_{n+1} - T_n \leq t / X_0, X_1, \ldots, X_n; T_0, T_1 \ldots, T_n\}$

= P$\{X_{n+1} = j, T_{n+1} - T_n \leq t / X_n\}$

$\forall$ n$\in N$,j $\in$ E and t $\in R_+$, 0=$T_0 \leq T_1 \leq \cdots$

Example: M/ G/1 Queue

It is a single server Queuing system subject to a Poisson process of arrivals with rate a and independent service times with the common distributions Φ .Let T_0=0, T_1, T_2 ..., be the instants of successive departures and let $X_n$be the no: of customers left behind by the n[th] departure .

Then (X,T) =$\{X_n, T_n : n \in N\}$ is a Markov Renewal process.

7. SEMI- MARKOV PROCESS

Let $N_j(t)$ be the no: of visits of the process to state j during (0,t] .

Define N(t) =$\sum_{j \in s} N_j(t)$

Let Y(t)= $Y_{N(t)}$ denote the state of the Markov Renewal process (X,T) . Then the stochastic process $\{Y_t : t \geq 0\}$ is called a Semi –Markov process .The vector process

N(t) = $\{N_0\,(t), N_1\,(t), \ldots\}$ may be identified as the Markov Renewal counting process.

Thus a MRP represents the transition epoch and the state of the process at that epoch. A Semi - Markov process represents the state of the MRP at an arbitrary time point.

Example: Open Network of Queues

Consider a Queuing systems with k-stations (servers) . A customer can join one of the station with probability $\{p_i\}$. He can move to j with probability p_{ij} after he visits i.

With probability $p_{i0} = 1 - \sum_j p_{ij}$ he can move out of the system. The service time in state I is assumed to have distributions with mean μ_i . Now we have a semi Markov process with state space (0,1,2,….k) with '0' as an absorbing state.

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