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# Tangent Frame Bundle (FM) and Differential Geometry

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*Abstract*— **In this paper we have to study the manifold with projective connection is called homogenous curvilinear co-ordinates we study the differential geometry of Frame bundle (FM) by using techinique of adapted frame of the infinitesimal variations of cross section of frame bundle FM with different connection .Also study the geometry of cross section of frame bundle FM of differentiable Manifold M.**

**The purpose of the present papers is to improve the the fact about fibred space in differential geometry with concept of frame bundle FM and Tensor calculus.**

*Keywords*— **Iron oxide, Iron oxyhydroxide, Nanowires, Structural and magnetic properties**

## I. INTRODUCTION

The tangent frame bundle (or simply the frame bundle) of a smooth manifold M is frame bundle associated to the tangent bundle of M.The frame bundle of M is often denoted FM or GL(M) rather than FM(TM).If M is n-dimensional then the tangent bundle has rank n,so the frame bundle of M is principal of GL(n,r) bundle over M.

The differential geometry of frame bundle FM was first extensively studied by T.Okubo[55],[56].Recently the lifts of different geometrical objects from manifold M to its frame bundle FM was studied by K.P. Mobs [46].

Coordinate systems in M are denoted by $(\mathrm{U}, x^i)$, where U is the coordinate neighbourhood and $x^i$ are coordinate functions. Components in $(\mathrm{U}, x^i)$ of geometric objection M will be referred to simple as components in U or just components. We denote the partial diffentiation $\frac{\partial}{\partial x^i}$ by $\partial_i$ and lie-derivative by $\wp_x$.

Let M be an n-dimensional differentiable man ifold o class $C^\infty$, denote by $T_x(M)$ the tangent space to M at a point $x \,\varepsilon\, M$. A frame $Z_x$ at x is an ordered basis $(x_1, x_2, \ldots\ldots x_n)$ of $T_x(M)$. Let FM be the set of all frames at all points of M. The natural projection $\pi : FM \to M$ is denoted by $Z_x \to x$. A $C^\infty$-structure can be introduced on FM as follows :

Let $(\mathrm{U}, x^i)$ be a coordinate system in M. Then $FU = \pi^{-1}(U)$ consists of all frames at all points of U. The vectors $X_\alpha$ of the frame $Z_x \,\varepsilon\, FU$ can uniquely expressed in the form $X_\alpha = X_\alpha^i \frac{\partial}{\partial x^i}$. So that $\{FU, (x^i, X_\alpha^i)\}$ is a coordinate system in FM. We call it induced coordinate system. Let $(U, x^i)$ and $(U^1, x^{i'})$ be two coordinate systems in M which are related by

(1.1) $\quad x^{i'} = x^{i'}(x^1, x^2, \ldots\ldots x^n)$

Then the induced coordinate systems $\{FU, (x^i, X_\alpha^i)\}$ and $\{FU', (x^{i'}, X_\alpha'^{i})\}$ are related by

(1.2) $\quad x^{i'} = x^{i'}(x^1, x^2, \ldots\ldots x^n)$

$$X_\alpha^{i'} = P_i^{i'} X_\alpha^i \quad \text{on } FU \cap FU'$$

Where, $P_i^{i'} = \frac{\partial x^{i'}}{\partial x^i}$.

From above equation, we furnish the conclusion that FM can be associated with the structure of an $(n+n^2)$ dimensional $C^\infty$ manifold. The natural projection $\pi : F_m \to M$ is than $C^\infty$. The general linear group GL (n, R) acts on FM to the right with $g = \left[g_{\alpha'}^{\alpha}\right]$ G $GL(n, R)$ taken the frame $Z = (X_\alpha)$ to the frame $Zg = (X_\alpha, g_{\alpha'}^{\alpha})$. The $g^\alpha$ translation Rg which maps Z to Zg is a homeomorphism of FM. It can be shown that FM is a principal fiber buble over M with $GL = (n, R)$ as the structure group[36]. Given this fibre bundle structure FM is called the frame bundle over M.

 

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The matrix $\left[X_{\alpha}^{i}\right]$ whose entries are the components of the frame $(X_{\alpha})$ is non singular. Its inverse will usually be written as $\left[X_{\alpha}^{i}\right]$. But sometimes, we also write $\left[X_{\alpha}^{i}\right]=\left[X_{j}^{i}\right]$ and $\left[X_{i}^{\alpha}\right]=\left[Y_{i}^{j}\right]$ which give alternative way of indicating the entries of these two matrices. We sometimes write $X^{i_\alpha}$ – in place of $X_{\alpha}^{i}$.

From 1.2 the following results are derived -

$$(1.3)\quad \begin{bmatrix} \frac{\partial}{\partial x^{i}}=P_{i}^{i'}\frac{\partial}{\partial x^{i'}}+P_{ij}^{i}X_{\alpha}^{i}\frac{\partial}{\partial X_{\alpha}^{i'}} \\ \frac{\partial}{\partial X_{\alpha}^{i}}=P_{i}^{i'}\delta_{\alpha}^{\alpha'}\frac{\partial}{\partial X_{\alpha'}^{i'}} \end{bmatrix}$$

on $FU \cap FU'$

Where, $P_{ij}^{i'} = \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j}$

Thus the Jacobean matrix of the transformation (1.12) in FM is

$$(1.4)\quad \begin{bmatrix} P_{i}^{i'} & 0 \\ P_{ij}^{i'}X_{\alpha}^{j} & P_{i}^{i'}\delta_{\alpha}^{\alpha'} \end{bmatrix}$$

## II. THE RIEMANNIAN METRICON FM

Let g be a Riemannian metric on M with natural components $g_{ji}$. Mok[46] has investigated that g can induce a Riemannian metric G on frame bundle FM whose line element in FU is given by

$$(2.1)\quad d\sigma^2 = g_{ji}\,dx^j\,dx^i + \delta_{\alpha\beta}\,g_{ji}\,\delta X_{\alpha}^{i}\,\delta X_{\beta}^{i}$$

Where, $\delta X_{\alpha}^{h}$ in the covariant differential equal to $dX_{\alpha}^{h}+\Gamma_{a}^{\alpha}dx^{i}$ by a simple computation the natural components $G_{AB}$ of G are found to be

$$(2.2)\quad G: G_{AB}\begin{bmatrix} g_{ji}+\delta_{\beta}^{\alpha}\,g_{mn}\,\Gamma_{j}^{m_\alpha}\,\Gamma_{i}^{h_\alpha} & g_{mg}\,\Gamma_{i}^{m_\alpha} \\ g_{mi}\,\Gamma^{m_\alpha} & g_{ji}\,\delta_{\alpha\beta} \end{bmatrix}$$

The frame components $G_{\lambda\mu}$ of G on FM are

$$(2.3)\quad G_{\lambda\mu}=\begin{bmatrix} g_{ji} & 0 \\ 0 & g_{ji}\,\delta\alpha\beta \end{bmatrix}$$

The frame components $\tilde{\Gamma}_{\mu\nu}^{\lambda}$ of the Riemannian connection of G on FM are given by Mok [46].

$$(2.4)\quad \tilde{\Gamma}_{ji}^{\lambda}=\Gamma_{ji}^{h};\quad \tilde{\Gamma}_{ji}^{h_\alpha}=-\frac{1}{2}R_{ji}^{h}\,X_{\alpha}^{a}$$

$$\tilde{\Gamma}_{ji_\alpha}^{h}=-\frac{1}{2}R_{j\alpha i}^{h}\,X\,,\quad \tilde{\Gamma}_{j\alpha i}^{h}=-\frac{1}{2}R_{j\alpha i}^{h}\,X_{\alpha}^{a}$$

$$\tilde{\Gamma}_{ji_\beta}^{h_\alpha}=\Gamma_{ji}^{h}\,\delta_{\beta}^{\alpha};\quad \tilde{\Gamma}_{j_\beta i}^{h_\alpha}=0;$$

$$\tilde{\Gamma}_{j_\alpha i_\beta}^{h}=0;\quad \tilde{\Gamma}_{j_\beta i_\alpha}^{h_\alpha}=0.$$

Let $\tilde{R}$ be the cavatina tensor of the metric tenser G on FM. Its frame components $\tilde{R}_{\omega\lambda\mu}^{\gamma}$ are given by Mok [46].

$$(2.5)\quad \tilde{R}_{k_{ji}}^{h}=R_{k_{ji}}^{h}+\frac{1}{4}\left(R_{k_{ba}}^{h}R_{j_{ic}}^{h}-R_{j_{ba}}^{h}R_{k_{ic}}^{a}\right.$$

$$X_{\varepsilon}^{b}X_{\varepsilon}^{c}-\frac{1}{2}R_{k_{ji}}^{a}=R_{k_{jb}}^{h}R_{i_{ca}}^{h}X_{\varepsilon}^{b}X_{\varepsilon}^{c};$$

$$\tilde{R}_{kji_\alpha}^{h}=\frac{1}{2}\left(\Delta_k R_{jia}^{h}-\Delta_j R_{kai}^{h}\right)\Big)X_{\alpha}^{a}$$

$$\tilde{R}_{k_{ji}}^{h_\alpha}=R_{kji}^{h}\,\delta_{\beta}^{\alpha}+\frac{1}{4}\left(R_{kab}^{h}R_{jci}^{a}-R_{jab}^{h}R_{kci}^{a}\right)X_{\delta}^{b}X_{\alpha}^{c}$$

$$\tilde{R}_{ki_{p}i}^{h_\alpha}=-\frac{1}{2}\left(\Delta_k R_{iaj}^{h}\right)X_{\beta j}^{a}$$

$$\tilde{R}_{kj\beta i_\alpha}^{h}=\frac{1}{2}R_{kji}^{h}\,\delta_{\alpha}^{\beta}-\frac{1}{4}R_{abj}^{h}R_{kci}^{a}X_{\beta}^{b}X_{\alpha}^{c}$$

$$\tilde{R}_{kj\beta i}^{h_\alpha}=\frac{1}{2}R_{kji}^{h}\,\delta_{\beta}^{\alpha}+\frac{1}{2}R_{kab}^{h}R_{icj}^{a}X_{\alpha}^{b}X_{\beta}^{c};$$

## III. THE CROSS-SECTION OF A FRAME BUNDLE F.M.

Let there be a global frame field $V_\alpha$ in an n-dimensional mainfold M, which is expressed locally in the form

$$v_\alpha = V_{\alpha}^{h}(x)\frac{\partial}{\partial x^h}$$

Then, the frame field $V_\alpha$ defines a Cross-section in the frame bundle FM, which is represented parametrically as

$$(3.1)\quad x^i = d^i,\quad X_\alpha(x)$$

with respect to induced coordinates $\left(x^i = x_{\alpha}^{i}\right)$ in FM.

By taking derivative of (1.3.1), we observe that there are n tangent vectors $B_i^A$ to cross-section of FM with natural components $B_i^A$ as

$$B_i^A = \begin{bmatrix} \delta_i^h \\ \delta_i, V^{h_\alpha} \end{bmatrix}, \text{ where } V^{h_\alpha} = V_\alpha^h \quad (3.2)$$

Again, on other hand, the fiber of the frame bundle FM is locally expressed as

$$x^h = \text{constant}; \quad X_\alpha = X_\alpha \quad (3.3)$$

The tangent vectors $C_{i_\alpha}^A$ to the fibre have natural components.

$$C^A : C_{i_\alpha}^A = \begin{bmatrix} 0 \\ \delta_i^h, \delta_p^\alpha \end{bmatrix} \quad (3.4)$$

Let U be a coordinate neighbourhood of M. Then in $\pi^{-1}(U)$, the $(n+n^2)$ local vector $B_i^A$ and $C_{i_\alpha}^A$ being linearly independent from a local family of frames along the cross-section. The co-frame $\left(B_A^h, C_A^{h_\alpha}\right)$ corresponding to their frame is given by

$$B_A^h = \left(\delta_i^h, 0\right) \quad (3.5)$$

$$C_A^{h_\alpha} = \left(\delta_i \;\; V^{h_\alpha}, \delta_i^h \, \delta_\beta^\alpha\right) \quad (3.6)$$

From above (3.3), we obtain the frame components $B_i^\mu$ of $B_i^A$ as

$$B_i^\mu = \begin{bmatrix} \delta_\lambda^h \\ \Delta_i, V^{h_\alpha} \end{bmatrix} \quad (3.7)$$

Similarly from above and (1.3.3) we obtain the following frame components $C_{i_\alpha}^\mu$ of $C_{i_\alpha}^\mu$,

$$C_{i_\alpha}^A = \begin{bmatrix} 0 \\ \delta_i^h, \delta_\beta^\alpha \end{bmatrix} \quad (3.8)$$

Moreover, it $B_\mu^h$ and $C_\mu^{h_\alpha}$ be the frame components of $B_A^h$ and $C_A^{h_\alpha}$ respectively, then we have

$$B_\mu^h = \left(\delta_i^h, \; 0\right) \quad (3.9)$$

$$C_\mu^{h_\alpha} = \left(\Delta_i, \; V^{h_\alpha}, \; \delta_i^h, \; \delta_p^\alpha\right) \quad (3.10)$$

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