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Numerical Solutions of Hybrid Fuzzy Fractional Differential Equations in a Hilbert Space

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Abstract— **The appearing of required differential equations is a significant problem in applied sciences and engineering, while the numerical method to show such a dynamical system is to use hybrid fuzzy fractional differential equations. In this paper, we study the numerical solutions of the hybrid fuzzy fractional differential equations by using the reproducing kernel Hilbert space method. The result depends on creating an orthogonal basis from the kernel functions and the solutions with arrangement structure regarding their r-cut representation in Hilbert space. Models are acquainted with a plot of the theory.**

Keywords— **Hybrid fuzzy fractional differential equations; Reproducing kernel Hilbert space method; Gram-Schmidt process**

I. INTRODUCTION

In expressing real physical phenomena, ordinary differential equations applied to many control areas, especially economics, material science, applied mathematics, and engineering. Most of the practical problems in question under differential meanings require solutions related to hybrid fuzzy fractional differential equations that meet the given initial conditions; therefore, given problems must be solved. We consider analytic and approximate solutions given with series form in terms of their parametric forms in the space $\bigoplus_{j=1}^{2} W_2^n$[a,b]. By using the following system of reproducing kernel Hilbert space (RKHS) method under the numerical solutions for hybrid fuzzy fractional differential equations (HFFDEs) of the general form:

$$D^{\beta}x(t) = f\big(t, x(t), \lambda_k(x_k)\big),\ \ t \in [t_k, t_{k+1}], \beta \in [0,1]$$
$$x(t_k) = x_k. \qquad \text{--------(1)}$$

Where

$0 \le t_0 < t_1 \dots\dots\dots < t_k \dots\dots\dots$, $f \in C[\Re^+ \times E \times E, E]$, $\lambda_k \in C[E, E]$.

The solutions of HFFDEs have a significant function in the field of science and engineering. To deal with this in more realistic circumstances, the HFFDEs are commonly solved using numerical methods. The RKHS strategy has been used effectively for a variety of notables applications in numerical analysis, computational mathematics, image processing, machine learning, probability, and statistics. The RKHS technique is a valuable structure for developing mathematical solutions of extraordinary enthusiasm to applied science [8]. Lately, [9,12,13,20] dependents on this theory, broad works have proposed and talked about for the numerical solutions of a few basic and differential operators next to each other with their hypothesis. Some new details about RKHS method, including its modification and scientific applications, its characteristics and kernel functions have been reported by authors [1-5,10,16-19]. This paper deals with the numerical solutions of hybrid fuzzy fractional initial value problems under generalized differentiability concept. Reproducing kernel Hilbert space method to obtain the existence of solutions to the hybrid fuzzy fractional initial value problem. Further, we show that numerical solutions of hybrid fuzzy fractional initial value problems obtained by iterative methods are more precise and concur well with specific solutions. Two examples introduce to verify the mathematical model of the proposed methods.

II. PRELIMINARIES

In this section, we present fuzzy analytical hypotheses and some essential definitions from the primary results. As for the obscure derivative concept, we will follow strongly generalized differentiability, which is a modification of the Hukuhara variant and has the advantage of dealing correctly with HFFDEs.

Definition 1:[11]

Let S be a nonempty set; a fuzzy set u in S is characterized by its membership function $u: S \to [0,1]$. Thus, $u(s)$ is the degree of membership of an element s in the fuzzy set u for each $s \in S$. A fuzzy set u on $\Re$ is called convex if $s, t \in \Re$ and $\lambda \in [0,1]$, $u(\lambda s + (1-\lambda)t) \ge min\{u(s), u(t)\}$ is called upper semi-continuous. If for each $r \in [0,1]$,$\{s \in \Re | u(s) \ge r\}$ is closed set, if $\{s \in \Re | u(s) = 1\}$ is called normal set, if $\{s \in \Re | u(s) > 0\}$ is support of a fuzzy set.

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Definition 2:[11]

Let u is a fuzzy number if and only if $[u]^r$ is compact convex subset of $\Re$ for $r \in [0,1]$ and $[u]^1 \neq \phi$. If u is a fuzzy number, then $[u]^r = [u_1(r), u_2(r)]$, for each $s \in [u]^r$, $r \in [0,1]$, where $u_1(r) = min\{s\}, u_2(r) = max\{s\}$ and $[u]^r$ is called $r-$cut representation form of a fuzzy number u.

Theorem 1: [11]

Let $u_1, u_2: [0,1] \to \Re$ satisfy the following conditions:

- u_1 is a bounded decreasing function,
- u_2 is a bounded increasing function,
- $u_1(1) \leq u_2(1)$,
- $lim_{r\to k^-} u_1(r) = u_1(k)$ and $lim_{r\to k^-} u_2(r) = u_2(k), k \in (0,1]$,
- $lim_{r\to 0^+} u_1(r) = u_1(0)$ and $lim_{r\to 0^+} u_2(r) = u_2(0)$.

Then $u: \Re \to [0,1]$, defined by $u(s) = sup\{r | u_1(r) \leq s \leq u_2(r)\}$ is a fuzzy number with parameter $[u_1(r), u_2(r)]$. Furthermore, if $u_1, u_2: [0,1] \to \Re$ is a fuzzy number with parameterize $[u_1(r), u_2(r)]$, then the functions u_1 and u_2 satisfy the aforementioned conditions.

Definition 3:[11]

The complete metric structure on $\Re_F$ is given by the Hausdorff distance mapping $D: \Re_F \times \Re_F \to \Re^+ \cup \{0\}$ such that $D(u,v) = sup_{\{0 \leq r \leq 1\}} max\{|u_{1r} - v_{1r}|, |u_{2r} - v_{2r}|\}$ for arbitrary fuzzy numbers u and v.

Definition 4:[4,6,11]

If u and v are two fuzzy numbers, for each $r \in [0,1]$, we've

- $[u+v]^r = [u]^r + [v]^r = [u_{1r} + v_{1r}, u_{2r} + v_{2r}]$,
- $[\lambda u]^r = \lambda [u]^r = [min\{\lambda u_{1r}, \lambda u_{2r}\}, max\{\lambda u_{1r}, \lambda u_{2r}\}]$,
- $[uv]^r = [u]^r[v]^r = [min\{u_{1r}v_{1r}, u_{1r}v_{2r}, u_{2r}v_{1r}, u_{2r}v_{2r}\}, max\{u_{1r}v_{1r}, u_{1r}v_{2r}, u_{2r}v_{1r}, u_{2r}v_{2r}\}]$,
- $u = v \; iff \; [u]^r = [v]^r \; iff \; u_{1r} = v_{1r}$ and $u_{2r} = v_{2r}$,

the collection of all fuzzy numbers with these addition and scalar multiplication is a convex cone.

Definition 5:[11]

Let u, v and $w \in \Re_F$, such that $u = v + w$; then w is called the Hukuhara differentiable of u and v, denoted by $u \ominus v$. Let $u \ominus v \neq u + (-1)v = u - v$ is Hukuhara differentiable, then $[u \ominus v]^r = [u_{1r} - v_{1r}, u_{2r} - v_{2r}]$.

Definition 6:[4,11]

Let x is strongly differentiable at $t_0 \in [a,b]$ and $x: [a,b] \to \Re_F$ such that

- For each $h > 0$, the Hukuhara differences $x(t_0 + h) \ominus x(t_0), x(t_0) \ominus x(t_0 - h)$ and

$$lim_{h\to 0^+} \frac{x(t_0+h) \ominus x(t_0)}{h} = lim_{h\to 0^+} \frac{x(t_0) \ominus x(t_0-h)}{h} = x'(t_0),$$

- For each $h < 0$, the Hukuhara differences $x(t_0) \ominus x(t_0 + h), x(t_0 - h) \ominus x(t_0)$ and

$$lim_{h\to 0^-} \frac{x(t_0+h) \ominus x(t_0)}{h} = lim_{h\to 0^-} \frac{x(t_0) \ominus x(t_0-h)}{h} = x'(t_0),$$

Theorem 2: [11]

Let $x: [a,b] \to \Re_F$ and put $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$, for each $r \in [0,1]$.

- If x is $(1)-$differentiable on $[a,b]$, then x_{1r} and x_{2r} are differentiable functions on $[a,b]$ and $[D_1 x(t)]^r = [x'_{1r}(t), x'_{2r}(t)]$,
- If x is $(2)-$differentiable on $[a,b]$, then x_{1r} and x_{2r} are differentiable functions on $[a,b]$ and $[D_2 x(t)]^r = [x'_{2r}(t), x'_{1r}(t)]$

Definition 7:[14,15]

Let $f \in L^F(I)$. The Riemann-Liouville fractional integral of order β of the fuzzy number valued function f is defined as follows:

$$J_a^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(\xi)}{(x-\xi)^{1-\beta}} d\xi \;, \; x > a$$

where $\Gamma(\beta)$ is the well-known Gamma function.

Definition 8:[14,15]

If $f \in AC(I)$, then Riemann-Liouville fractional derivative of order β of the crisp function f exists almost every where on I and can be represented by

$$^{RL}_{a}D^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x f(\xi)(x-\xi)^{-\beta} d\xi$$

Note that Riemann-Liouville fractional derivative of order β of f is the first order derivative of the fractional integral $1-\beta$ of f.

Definition 9:[14,15]

If $f \in AC(I)$, then Caputo fractional derivative of order β of the crisp function f exists almost everywhere on I and can be represented by

$$^{C}_{a}D^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x f'(\xi)(x-\xi)^{-\beta} d\xi$$

Note that Caputo fractional derivative of order β of f is the fractional integral $1-\beta$ of the first order derivative of f.

Definition 10:[14]

Let $f \in AC^F(I)$ and $G(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x f(\xi)(x-\xi)^{-\beta} d\xi$, for $x > a$. If the fuzzy number valued function G is $(1)-$differentiable, then Riemann-Liouville fractional derivative of order β of the fuzzy number valued function f exists almost everywhere on I and can be represented by ${}^{RL}_{a}D_1^{\beta} f(x) = \frac{d}{dx} G(x)$. If the fuzzy number valued function G is $(2)-$differentiable, then Riemann-Liouville fractional derivative of order β of the fuzzy number valued function f exists almost everywhere on I and can be represented by ${}^{RL}_{a}D_2^{\beta} f(x) = \frac{d}{dx} G(x)$.

Definition 11:[7]

Let H be a Hilbert space of function $f: E \to F$ on a set E, where E be a nonempty abstract set. A function $K: E \times E \to C$ is a reproducing kernel of Hilbert space H if the following conditions are satisfied:

- For each $t \in E; K(.,t) \in H$,
- For each $t \in E$ and $f \in H$; $< f, K(.,t) >_H = f(t)$.

The condition 2 is called "the reproducing property" which means that the value of a function f at a point t is reproducing by the inner product of f with $K(.,t)$.

Definition 12:[16]

The inner product space $W_2^1\,[a,b]$ is defined as $W_2^1\,[a,b] = \{z(t): z(t)$ is absolutely continuous real-valued functions on $[a,b]$, $z'\{(t)\} \in L^2[a,b]$, and $z(a) = 0\}$. The inner product and the norm in $W_2^1\,[a,b]$ are given by $< z_1(t), z_2(t) > W_2^1 = z_1(a)z_2(a) + \int_a^b z'_1(t)z'_2(t)dt$. Here $z_1, z_2 \in W_2^1\,[a,b]$ and $||\, z_1||_{W_2^1} = \sqrt{< z_1(t), z_2(t) > W_2^1}$

III. THEORY OF HYBRID FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

Consider the hybrid fuzzy fractional differential equations,

$$D^{\beta} x(t) = f\big(t, x(t), \lambda_k(x_k)\big),\ t \in [t_k, t_{k+1}],\ \beta \in [0,1]$$
$$x(t_k) = x_k, \qquad \text{------------(2)}$$

where $0 \le t_0 < t_1 < \cdots < t_k < \cdots, f \in C[\Re^+ \times E \times E, E]$, $\lambda_k \in C[E, E]$. The existence and uniqueness of solutions of the hybrid system hold on each $[t_k, t_{k+1}]$. A hybrid fractional differential equation initial value problem based on equation (1) is a system of fractional differential equation initial value problems of the form

$$D^{\beta} x(t) = \begin{cases} D^{\beta} x_0(t) = f(t, x_0(t), \lambda_0(x_0)), & x_0(t_0) = x_0,\ t_0 \le t \le t_1 \\ D^{\beta} x_1(t) = f(t, x_1(t), \lambda_1(x_1)), & x_1(t_1) = x_1,\ t_1 \le t \le t_2 \\ \vdots & \\ D^{\beta} x_k(t) = f(t, x_k(t), \lambda_k(x_k)), & x_k(t_k) = x_k,\ t_k \le t \le t_{k+1}, \end{cases} \qquad \text{--------------(3)}$$

By the solutions of (2) we mean the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), & t \in [t_0, t_1], \\ x_1(t), & t \in [t_1, t_2], \\ \vdots & \\ x_k(t), & t \in [t_k, t_{k+1}], \\ \vdots & \end{cases} \qquad \text{--------------(4)}$$

We note that the solutions of equation (2) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E$ and $k = 0,1,2,\cdots$. These formulation of equation (1) is first calculated as set of differential equation subject to set of initial conditions, next to apply RKHS algorithm, we get the function $x(t)$ in $r-$cut form $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$ and $[x(a)]^r = [\beta_{1r}, \beta_{2r}]$. By consider the parametric form of HFFDE (1), we've

$$D^{\beta}\,[x(t)]^r = [f\big(t, x(t), \lambda_k(x_k)\big)]^r, \quad t \in [t_k, t_{k+1}],\ \beta \in [0,1]$$
$$[x(t_k)]^r = [x_k]^r, \qquad \text{------------(5)}$$

where the endpoint functions of $[f\big(t, x(t), \lambda_k(x_k)\big)]^r$ are given by the set $[f\big(t, x(t), \lambda_k(x_k)\big)]^r = [f_{1r}\big(t, x(t), \lambda_k(x_k)\big),\ f_{2r}\big(t, x(t), \lambda_k(x_k)\big)$.These formulation of equation (5) together with the characteristic Theorems (1) and (2) show us how to deal with numerical solutions of HFFDEs.

To find the solutions of HFFDE (1), we've the following steps:

If $x(t)$ is differentiable, then we use $[Dx(t)]^r$ and solve HFFDE (1) translates into the following system

- Solve the equations for $x(t)$

$$D^{\beta}\, x_{1r}(t) = f_{1r}\big(t, x(t), \lambda_k(x_k)\big),$$
$$D^{\beta}\, x_{2r}(t) = f_{2r}\big(t, x(t), \lambda_k(x_k)\big). \text{-----(6)}$$

  subject to the initial conditions

$$x_{1r}(t_k) = x_{1rk}\ ,$$
$$x_{2r}(t_k) = x_{2rk}\ . \qquad \text{---------(7)}$$

- Ensure that $[x_{1r}(t_k), x_{2r}(t_k)]$ and $[D^{\beta}\, x_{1r}(t), D^{\beta}\, x_{2r}(t)]$ are valid sets for every $r \in [0,1]$,
- Calculate the solution $x(t_k)$ such that $[x(t_k)]^r = [x_{1r}(t_k), x_{2r}(t_k)]$.

IV. REPRODUCING KERNEL HILBERT SPACE METHOD

To apply the RKHS method, we defined linear operator $L: W[a,b] \to H[a,b]$ such that $L\, x_r(t) = D^{\beta}\, x_r(t)$. Put $f_r = (f_{1r}, f_{2r})^T$, $x_r = (x_{1r}, x_{2r})^T$, $L = diag(L_1, L_2)$. The system of equations can be converted into the equivalent form as:

$$L\, x_r(t) = f_r\big(t, x_r(t), \lambda_k(x_k)\big)$$
$$x(t_k) = x_k \qquad \text{---(8)}$$

where $x_r \in W[a,b]$ and $f_r \in H[a,b]$.

Next, an orthogonal function of $W[a,b]$: put $\varphi_{ij}(t) = H(t)e_j$ and $\psi_{ij}(t) = L^*(t)\varphi_{ij}(t)$, $i = 1,2,3,\cdots, j = 1,2$ where $e_1 = (1,0)^T$, $e_2 = (0,1)^T$, $L^* = diag(L_1^*, L_2^*)$ is the adjoint operator of L, $H(t)$ is the reproducing kernel function of $W[a,b]$, and $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$. The Gram-Schmidt orthogonal process,

$$\bar{\psi}_{ij}(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\beta_{ij}\ \psi_{ij}(t), \qquad \text{----------------(9)}$$

where β_{ij} are orthogonal coefficient and the following conditions:

- For $i = j = 1$, $\beta_{ij} = \frac{1}{||\psi_{11}||}$,
- For $i = j \neq 1$, $\beta_{ij} = \frac{1}{\sqrt{||\psi_{ij}||^2 - \sum_{p=1}^{i-1} <\psi_{ij}(t), \bar{\psi}_{ij}(t)>^2}}$,
- For $i > j$, $\beta_{ij} = \frac{-1}{\sqrt{||\psi_{ij}||^2 - \sum_{p=1}^{i-1} <\psi_{ij}(t), \bar{\psi}_{ij}(t)>^2}} \sum_{p=j}^{i-1} < \psi_{ij}(t), \bar{\psi}_{ij}(t) > \beta_{pj}$

Theorem 3:

The operator $L: W_2^{n+1}[a,b] \to W_2^1[a,b]$ are bounded and linear.

Proof:

1. The linear part is obvious.
2. The bounded part first we get to prove that $||L\, x_r||^2 \leq M_k||\, x_r||^2$, where $M_k > 0$

by the inner product norm, we get $||L\ x\_r||^2 = \ < L\ x_r(t), L\ x_r(t) > = \int_a^b \{[(L\ x_r)'(t)]^2 + [(L\ x_r)(t)]^2\}\, dt$. By reproducing property of $R(x,y)$, we have $x_r(t) =< x_r(t), R(x,y) >$, $(L\ x_r)(t) =< x_r(t), L\, R(x,y) >$ and $(L\ x_r)'(t) =< x_r(t), (L\, R(x,y))' >$. By Schwarz inequality, we've

$$|(L\ x_r)(t)| \ = |< x_r(t), L\, R(x,y) >| \leq ||L\, R(x,y)||||\, x_r|| = M_{k1}||\, x_r||,$$
$$|(L\, x_r)'(t)| \ = |< x_r(t), (L\, R(x,y))' >| \leq ||(L\, R(x,y))'||||\, x_r|| = M_{k2}||\, x_r||,$$

where $M_{k1}, M_{k2} > 0$. Thus, $||L\ x_r||^2 \leq ((M_{k1})^2 + (M_{k2})^2)\,(b-a)\,||\, x_r||^2$ or $||L\ x_r|| \leq M_k||\, x_r||$, in which $M_k = \sqrt{((M_{k1})^2 + (M_{k2})^2)(b-a)}$.

Similarly, the operator $L: W_2^{\{n+1\}}[a,b] \to W_2^1[a,b]$, such that $L\, x(t) = D^{\beta}x(t) = f\big(t, x(t), \lambda_k(x_k)\big)$. It is clear that L is a bounded linear operator from $W_2^{\{n+1\}}[a,b] \to W_2^1[a,b]$.

Remark 1:

The operator $L: W[a,b] \to H[a,b]$ is bounded and linear.

Remark 2:

For equation (8), if $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then $\{\psi_{ij}(t)\}_{i=1,j=1}^{\infty,2}$ is the complete function system of $W[a,b]$ and $\{\psi_{ij}(t)\} = L\, R(x,y)$.

Theorem 4:

If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of equation (8) is unique, then the analytic solution of equation (8) satisfies the form

$$x_r\ (t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2}\beta_{ij}\ f_r\big(t, x_r(t), \lambda_k(x_k)\big)\,\bar{\psi}_{ij}(t). \text{---(10)}$$

Proof: Apply remark (2), given result $\{\psi_{ij}(t)\}_{i=1,j=1}^{\infty,2}$ is the complete orthonormal basis of $W[a,b]$. For all $x_r(t) \in W[a,b]$, and $\sum_{i=1}^{\infty}\sum_{j=1}^{2} < x_r(t), \bar{\psi}_{ij}(t) > \bar{\psi}_{ij}(t)$ is the Fourier series, then the series is convergent in the sence $||.||$. Thus, using equation (11), we've

$$\begin{aligned} x_r\ (t) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} < x_r(t), \bar{\psi}_{ij}(t) >_W\ \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} < x_r(t), \beta_{ij}\psi_{ij}(t) >_W\ \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\beta_{ij} < x_r(t), L^*\varphi_{ij}(t) >_H\ \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\beta_{ij} < Lx_r(t), \varphi_{ij}(t) >_H\ \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\beta_{ij} < f_r\big(t, x_r(t), \lambda_k(x_k)\big), \varphi_{ij}(t) >_H\ \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\beta_{ij}\ f_r\big(t, x_r(t), \lambda_k(x_k)\big)\,\bar{\psi}_{ij}(t) \end{aligned}$$

therefore, equation (10) is analytic solution of equation (8).

Remark 3:

The approximate solution $x_r^n(t)$ for $x_r(t)$ of equation (8) have finitely many terms of $x_r(t)$ for equation (10) and is given by

$$x_r^n(t) = \sum_{i=1}^{n}\sum_{j=1}^{2}\beta_{ij}\ f_r\big(t, x_r(t), \lambda_k(x_k)\big)\,\bar{\psi}_{ij}(t). \text{----(11)}$$

V. CONSTRUCTION OF ITERATIVE METHOD

If the equation (8) is non linear, then the analytic solutions can be calculated using the following iterative method

$$x_r(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} L_{ij}\ \bar{\psi}_{ij}(t) \qquad \text{-----------------(12)}$$

where $L_{ij} = \beta_{ij}f_r\big(t, x_r(t), \lambda_k(x_k)\big)$. For numerical computations, we put the initial condition and n-term by

$$x_r{}^n(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\ \bar{\psi}_{ij}(t) \qquad \text{-------------------(13)}$$

where the coefficients B_{ij} of $\bar{\psi}_{ij}(t)$, $i = 1,2,\cdots n, j = 1,2$ are given below

$$B_{1j} = \sum_{i=1}^{1}\sum_{j=1}^{2}\beta_{ij}{}^1\ f\big(t, x_r(t), \lambda_k(x_k)\big);$$
$$x_r{}^1(t) = \sum_{i=1}^{1}\sum_{j=1}^{2} B_{ij}\ \bar{\psi}_{ij}(t),$$
$$B_{2j} = \sum_{i=1}^{2}\sum_{j=1}^{2}\beta_{ij}{}^2\ f\big(t, x_r(t), \lambda_k(x_k)\big);$$
$$x_r{}^2(t) = \sum_{i=1}^{2}\sum_{j=1}^{2} B_{ij}\ \bar{\psi}_{ij}(t),$$
$$\vdots$$
$$B_{nj} = \sum_{i=1}^{n}\sum_{j=1}^{2}\beta_{ij}{}^n\ f\big(t, x_r(t), \lambda_k(x_k)\big);$$
$$x_r{}^n(t) = \sum_{i=1}^{n}\sum_{j=1}^{2} B_{ij}\ \bar{\psi}_{ij}(t), \text{-----------------------(14)}$$

In the iterative process of condition (14), we can ensure that the guess ${x_r}^n(t)$ satisfies the underlying state of condition (8). Presently, we will demonstrate that ${x_r}^n(t)$ in the iterative recipe of condition (14) is merged to the logical arrangement $x_r(t)$ of condition (8). In reality, this result is an essential standard in the RKHS hypothesis and its applications.

Remark 4:

If $z(t) \in W_2^{\{n+1\}}[a,b]$, then $|z^{\{(n-1)\}}(t)\ | \le \ (1+b-a+\sqrt{\{(b-a)^3\}}\,)\ ||z||_{\{W_2^{\{n+1\}}\}}$ and $\left|z^n(t)\ \right| \le \left(1+\sqrt{\{b-a\}}\right)\left|\,|z|\right|_{W_2^{n+1}}$

If $\left||x^n - x\right|| \to 0,\ t_k \to s$ as $n \to \infty$, $\left||x^n\right||$ is bounded, and $f\left(t, x_r(t), \lambda_k(x_k)\right)$ is continuous, then $f\left(t_k, x^n(t_k), \lambda_k(x_k)\right) \to f\left(s, x(s), \lambda_k(x_k)\right)\ as\ n \to \infty.$

Remark 5:

Suppose that $\left||x^n\right||$ is bounded in equation (13) and equation (8) has a unique solution. If $\{t_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then the n −term approximate solution $x^n(t)$ in the iterative method of equation (13) converges to the analytic solution $x(t)$ of (8) and $x(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2} B_{ij}\ \ \bar{\psi}_{ij}(t)$, where B_{ij} are given by equation (14).

Theorem 5:

Let $\varepsilon_n = ||x - x^n||$, where $x(t)$ and $x^n(t)$ are given by equations (8) and (9) respectively. Then, the sequence of numbers $\{\varepsilon_n\}$ are monotone decreasing in the sense of the norm of $W[a,b]$ and $\varepsilon_n \to\ 0$ as $n \to\ \infty$.

Proof: From remark (5), it is obvious that

$$\varepsilon^2{}_n = \left|\left|\sum_{i=n+1}^{\infty}\sum_{j=1}^{2} < f\left(t, x(t), \lambda_k(x_k)\right),\ \bar{\psi}_{ij}(t) >_W\ \bar{\psi}_{ij}(t)\right|\right|^2$$

$$= \sum_{i=n+1}^{\infty}\sum_{j=1}^{2} < f\left(t, x(t), \lambda_k(x_k)\right),\ \bar{\psi}_{ij}(t) >^2{}_W$$

$$\varepsilon^2{}_{n-1} = \left|\left|\sum_{i=n}^{\infty}\sum_{j=1}^{2} < f\left(t, x(t), \lambda_k(x_k)\right),\ \bar{\psi}_{ij}(t) >_W\ \bar{\psi}_{ij}(t)\right|\right|^2$$

$$= \sum_{i=n}^{\infty}\sum_{j=1}^{2} < f\left(t, x(t), \lambda_k(x_k)\right),\ \bar{\psi}_{ij}(t) >^2{}_W$$

Clearly, $\varepsilon_{n-1} \ge\ \varepsilon_n$, and consequent of $\{\varepsilon_n\}$ are monotone decreasing in the sense of the norm of ‖.‖. By Theorem (4), $\sum_{i=n}^{\infty}\sum_{j=1}^{2} < f\left(t, x(t), \lambda_k(x_k)\right),\ \bar{\psi}_{ij}(t) >_W\ \ \bar{\psi}_{ij}(t)$ is convergent;

$\varepsilon^2{}_n = \sum_{i=n+1}^{\infty}\sum_{j=1}^{2} < f\left(t, x(t), \lambda_k(x_k)\right),\ \bar{\psi}_{ij}(t) >^2{}_W\ \to 0$

Or $\varepsilon_n \to 0$ as $n \to \infty$.

VI. NUMERICAL EXAMPLES

Example 1:

Consider the initial value problem,

$$\begin{cases} D^{\beta}x(t) = x(t) + m(t)\lambda_k\left(x(t_k)\right),\ t \in\ [t_k, t_k+1], \\ t_k = k,\ k = 0,1,2,\cdots, \beta\ \epsilon\ [0,1] \\ x(0) = [0.75,1,1.125], \end{cases} \qquad \text{--------------(15)}$$

where

$$m(t) = \begin{cases} 2(t(mod 1)),\ if\ \ t(mod1) \le 0.5 \\ 2(1 - t(mod1)), if\ \ t(mod1) >\ 0.5, \end{cases} \qquad \text{------(16)}$$

$$\lambda_k(\mu) = \begin{cases} 0, & if \quad k = 0 \\ \mu,\ if & k \epsilon\{\,1,2,\cdots\} \end{cases} \qquad \text{----------(17)}$$

For [0,1], the exact solution of (15) satisfies, $x(t) = [0.75e^t, e^t, 1.125e^t]$.

For [1,1.5], the exact solution of (15) satisfies, $x(t) = x(1)\left(3e^{\{t-1\}} - 2t\right)$.

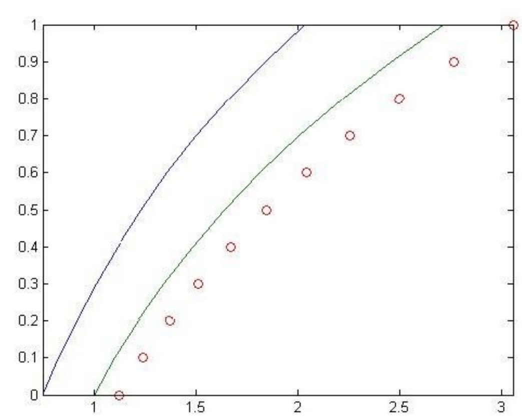
For [1.5,2], the exact solution of (15) satisfies, $x(t) = x(1)(2t - 2 + e^{\{t-1.5\}}(3\sqrt{e} - 4))$.

The numerical results of **Example 1** for various t in [0,1], $n = 10$ and $\beta = 1$ are shown in **Table 1** and **Fig 1**.

Table 1

t	Exact solution	Approximate solution
0	[0.750,1.000,1.125]	[0.750000000,1.000000000,1.125000000]
0.25	[0.963,1.284,1.445]	[0.963019062,1.284025416,1.444528593]
0.5	[1.236,1.648,1.855]	[1.236540953,1.648721270,1.854811429]
0.75	[1.588,2.117,2.382]	[1.587750012,2.117000016,2.381625018]
1	[2.039,2.718,3.058]	[2.038711371,2.718281828,3.058067057]

Fig 1: value of x(t)



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Example 2:

Consider the hybrid fuzzy fractional differential equation,
Consider the initial value problem,

$$\begin{cases} D^{\beta}x(t) = x(t) + m(t)\lambda_k\big(x(t_k)\big),\ t \in\ [t_k, t_k + 1], \\ \quad t_k = k,\ k = 0,1,2,\cdots, \beta\ \epsilon\ [0,1] \\ \quad x(0) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], \\ \quad 0 \le \alpha \le 1, \end{cases} \quad \text{-------(18)}$$

where

$$m(t) = \begin{cases} 2(t(mod1)),\ if\ \ t(mod1) \le 0.5 \\ 2(1 - t(mod1)), if\ \ t(mod1) >\ 0.5, \end{cases} \quad \text{-----(19)}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, \quad if \quad k = 0 \\ \mu,\ if\ \ k\epsilon\{1,2,\cdots\} \end{cases} \quad \text{----------(20)}$$

For [0,1], the exact solution of (18) satisfies, $x(t) = [(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t]$.

For [1,1.5], the exact solution of (18) satisfies, $x(t) = x(1;\alpha)\big(3e^{\{t-1\}} - 2t\big)$.

For [1.5,2], the exact solution of (18) satisfies, $x(t) = x(1.5;\alpha)(2t - 2 + e^{\{t-1.5\}}(3\sqrt{e} - 4))$.

Where $x(1;\alpha) = [(0.75 + 0.25\alpha)e, (1.125 - 0.125\alpha)e]$

$x(1.5;\alpha) = x(1;\alpha)(\,3\sqrt{e} - 4)$

The numerical results of **Example 2** for various α in [0,1], $n = 10$, $\beta = 1$ and $t = 0.1,0.2,0.3,0.4$ are shown in **Table 2** and **Fig 2** and **Fig 3**. The numerical results of **Example 2** for various β in [0,1], $n = 10$, $\alpha = 1$ and $t = 0.1,0.2,0.3,0.4$ are shown in **Table 3.**

Table 2:

α	t=0.1	t=0.2	t=0.3	t=0.4
0	[0.83,1.24]	[0.92,1.37]	[1.01,1.52]	[1.12,1.68]
0.25	[0.89,1.21]	[0.99,1.34]	[1.09,1.47]	[1.21,1.63]
0.5	[0.97,1.17]	[1.06,1.29]	[1.18,1.43]	[1.31,1.58]
0.75	[1.04,1.14]	[1.14,1.26]	[1.26,1.39]	[1.39,1.54]
1	[1.11,1.11]	[1.22,1.22]	[1.35,1.35]	[1.49,1.49]

Fig 2: $\boldsymbol{\alpha = 0\ to\ 0.75}$

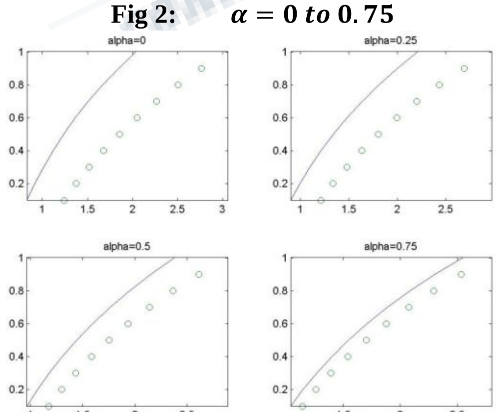

Fig 3: $\boldsymbol{\alpha = 1}$

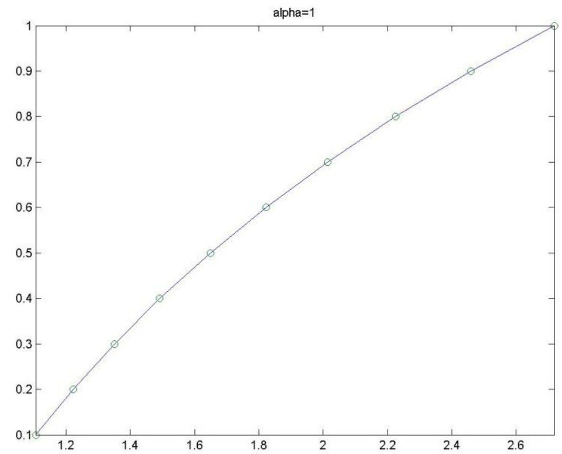

Table 3:

T	$\beta = 1/3$	$\beta = 1/2$	$\beta = 1$
0.1	[1.18,1.18]	[1.19,1.19]	[1.11,1.11]
0.2	[1.26,1.26]	[1.25,1.25]	[1.22,1.22]
0.3	[1.35,1.35]	[1.31,1.31]	[1.35,1.35]
0.4	[1.45,1.45]	[1.38,1.38]	[1.49,1.49]

VII. CONCLUSION

The use of HFFDEs is a method of requesting properties to show real miracles under potential vulnerabilities. In this study, we introduced another method for understanding HFFDEs to the application of the reproducing kernel theory. We have demonstrated the ability of the RKHS technique to predict solutions of HFFDEs under strongly generalized variability. By differentiating the results obtained using results and other significant mathematical or analytical methods, the proposed new strategy provides more accurate approximations, especially in nonlinear cases. Besides, after settling HFFDE, the arrangement is reachable at any subjective point in period time between work focuses. The fundamental purpose behind utilizing the RKHS method was their appropriateness in capacities estimate. Results got to show that the mathematical plan is viable and helpful for solving the HFFDE equation (1). Moreover, we note that a computational strategy introduced well demonstrated that the mistake of the rough arrangements is monotone decreasing in the feeling of the standard of W[a,b].

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