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On RHF and Bernoulli Polynomial for the numerical solution of differential equations

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***Abstract*— Approximation of the solution of the differential equations is done by Bernoulli polynomial. Bernoulli polynomial and operational matrix of differentiation were used in reducing differential equations into algebraic equations. The method and its application is demonstrated through illustrative examples and found that the method is computationally attractive. The Bernoulli polynomial method has been applied to compare the numerical solution of differential equations with the existing method of Rationalized Haar Function.**

***Index Terms*— Bernoulli polynomial, Rationalized Haar function(RHF), Differential equations**

I. INTRODUCTION

Differential equations have lots of application in both the science and engineering field. Differential equations (DEs) are used in different fields of mathematical modeling. Regularly, obtaining an analytical solution for some DEs is not possible. Thus, few numerical techniques were introduced to calculate approximate solutions for such equations. Such as Legendre polynomial[1], Chebyshev polynomial [2], Hermite polynomial [3,4], Bernoulli polynomial [5, 6]. Recently, a new method developed to solve numerical problems by the concept of graph theory called Hosoya polynomial, one can refer for graph theory terminologies and developed method in [7, 8, 9, 10, 11, 12,13]. Wavelet based numerical method, such as Modified wavelet full-approximation scheme [14], Bernoulli wavelet [15], Hermite wavelet [16] and Rationalized haar functions[17]. Bernoulli polynomial is applied for the numerical solution for integral equations [18]. This article, gives the Bernoulli polynomial method for the numerical solution of differential equations and comparison with the existing method(RHF)[17].

II. PROPERTIES OF BERNOULLI POLYNOMIAL AND FUNCTION APPROXIMATION

Bernoulli polynomial is defined by [5, 6],

$$B_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i$$

Where $\alpha_i,\ i = 0,1,\dots,m$ are Bernoulli numbers which are in a sequence of signed rational numbers emerging in the series expansion of trigonometric functions given by,

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The sequence of Bernoulli numbers is

$$\alpha_0 = 1, \alpha_1 = \frac{-1}{2}, \alpha_2 = \frac{1}{6}, \alpha_4 = \frac{-1}{30}, \alpha_6 = \frac{1}{42}, \alpha_8 = \frac{-1}{30}, \alpha_{10} = \frac{5}{66}, \ \dots$$

and $\alpha_{2i+1} = 0, i = 1,2,3,\dots$

The Bernoulli Polynomials are,

$$B_0(t) = 1, \quad B_1(t) = t - \frac{1}{2}, \quad B_2(t) = t^2 - t + \frac{1}{6},$$
$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \quad B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30},$$
$$B_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t, \quad B_6 = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 + \frac{1}{42},$$ and so on.

Function approximation: A function $f(x) \in L^2[0,1]$ is expanded as:

$$f(x) = \sum_{i=0}^{N} a_i B_i(t) = A^T B(t) \tag{2.1}$$

where A and $B(t)$ are $N \times 1$ matrices given by:

$$A = [a_0, a_1 \dots, a_N]^T \tag{2.2}$$

and

$$B(t) = [B_0(t), B_1(t), \dots, B_N(t)]^T. \tag{2.3}$$

1. Method of Solution

Here, let us take the differential equation

$$g_0(t)y''(t) + g_1(t)y'(t) + g_2(t)y(t) = g_3(t),\ t \in [0,1) \tag{3.1}$$

with initial condition $y(0) = y_0,\ y'(0) = y_0'$ (3.2)

where $g_0(t), g_1(t), g_2(t)$ and $g_3(t)$ are functions of a time variable t $(0 \le t < 1)$.

The approximate solution of function $y(t)$ using Bernoulli polynomials method is give by,

Step 1: Approximate $y(t)$ as truncated series given in Eqn. (2.1).

$$y(t) = A^T B(t) = \sum_{i=0}^{N} a_i B_i(t) \qquad (3.3)$$

Further, differentiate Eqn. (3.3)

$$y'(t) = A^T B'(t) = \sum_{i=0}^{N} a_i B_i'(t) \qquad (3.4)$$

$$y''(t) = A^T B''(t) = \sum_{i=0}^{N} a_i B_i''(t) \qquad (3.5)$$

Step 2: Substitute Eqn. (3.3), (3.4) & (3.5) in (3.1), we obtain

$$g_0(t)\left\{\sum_{i=0}^{N} a_i B_i''(t)\right\} + g_1(t)\left\{\sum_{i=0}^{N} a_i B_i'(t)\right\} + g_2(t)\left\{\sum_{i=0}^{N} a_i B_i(t)\right\} = g_3(t), \qquad (3.6)$$

Step 3: Substituting the collocation point $t_i = \frac{1}{2}\left(1 + \cos\left(\frac{(i-2)\pi}{N-1}\right)\right),\ i = 2,3,\ldots,N$ in Eqn. (3.6),

$$g_0(t)\left\{\sum_{i=0}^{N} a_i B_i''(t_i)\right\} + g_1(t)\left\{\sum_{i=0}^{N} a_i B_i'(t_i)\right\} + g_2(t)\left\{\sum_{i=0}^{N} a_i B_i(t_i)\right\} = g_3(t), \qquad (3.7)$$

Step 4: Using the initial condition $y(0) = y_0$ & $y'(0) = y_0'$ at $t = 0$ obtained the first two system of equations with unknown remaining system of equations is obtained by Eqn. (3.7). This gives the number of system of linear or nonlinear equations with unknowns.

Step 5: Solving these system of equations by using the Newton iterative scheme, we obtain the unknown Bernoulli coefficients 'A', substitute in (3.3), which gives the required approximate solution of (3.1).

III. Numerical Results

Here, we consider a few illustrative examples from the literature to test the accuracy and efficiency of the results:

$$Error\ function = \left\| y_e(t_i) - y_a(t_i) \right\|_\infty = \sqrt{\sum_{i=1}^{n} \left[y_e(t_i) - y_a(t_i) \right]^2}$$

where, y_e is exact solution and y_a is approximate solution.

Here, comparison of the numerical solutions with exact solutions and the existing method is Rationalized Haar functions (RHF) [17].

Illustration 1. Consider linear differential equations,

$$y''(t) + 2ty'(t) = 0, \quad t \in [0,1) \qquad (4.1)$$

with initial condition $y(0) = 0, \quad y'(0) = \frac{2}{\sqrt{\pi}}$

and the exact solution is $y(t) = \frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}\, erf(t)}{2}\right)$. By the proposed technique, solving the Eqn. (4.1) with initial condition is reduced into system of algebraic equations and the simplification gives the required approximate solution which is compared with the existing method(RHF). Table 1 gives the comparison of the present method with existing method. Fig. 1 shows the numerical solution with exact solution

Table 1. Comparison of present method with existing method(RHF)

t	Exact	RHF at $k = 8$	BPM at $N = 8$	Error(RHF)	Error(BPM)
0	0	0	0	0	0
0.1	0.112463	0.11244	0.112461	2.29E-05	1.44E-06
0.2	0.222703	0.22268	0.222701	2.26E-05	1.78E-06
0.3	0.328627	0.32861	0.328624	1.68E-05	2.26E-06
0.4	0.428392	0.42837	0.428389	2.24E-05	3.79E-06
0.5	0.5205	0.52047	0.520495	2.99E-05	5.03E-06
0.6	0.603856	0.60384	0.603851	1.61E-05	5.17E-06
0.7	0.677801	0.67779	0.677796	1.12E-05	5.16E-06
0.8	0.742101	0.74208	0.742095	2.10E-05	6.13E-06
0.9	0.796908	0.79689	0.796901	1.82E-05	7.08E-06
1	0.842701	0.84269	0.842694	1.08E-05	6.98E-06

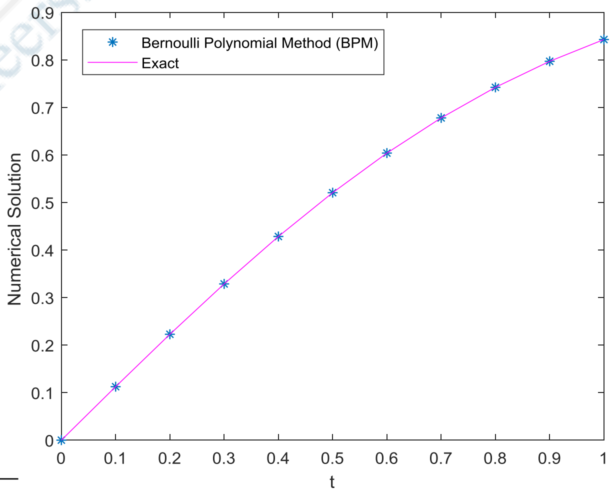

Figure 1: Numerical solution with exact solution

Illustration 2. Consider,

$$y''(t) + 2ty'(t) + y^3(t) = 2 + 4t^2 + t^6, \quad t \in [0,1) \qquad (4.2)$$

with initial condition $y(0) = 0, \quad y'(0) = 0$

and the exact solution is $y(t) = t^2$. By the proposed technique, solving the Eqn. (4.2) with initial condition is reduced into system of algebraic equations. On simplifying, which we get the required exact solution which is compared with the existing method(RHF). Table 2 gives the comparison of the present method with existing method. Fig. 2 shows the numerical solution with exact solution. This shows the efficiency of the proposed method is very accurate.

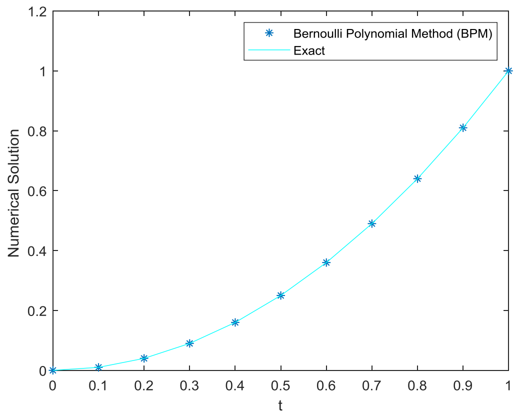

Figure 2: Numerical solution with exact solution

Table 2. Comparison of present method with existing method(RHF)

t	Exact	RHF at k = 8	BPM at N = 5	Error(RHF)	Error(BPM)
0	0	0	0	0	0
0.1	0.01	0.01002	0.01	2.00E-05	0
0.2	0.04	0.04003	0.04	3.00E-05	0
0.3	0.09	0.09002	0.09	2.00E-05	0
0.4	0.16	0.16003	0.16	3.00E-05	0
0.5	0.25	0.25002	0.25	2.00E-05	0
0.6	0.36	0.36002	0.36	2.00E-05	0
0.7	0.49	0.49003	0.49	3.00E-05	0
0.8	0.64	0.64001	0.64	1.00E-05	0
0.9	0.81	0.81	0.81	0	0
1	1	1	1	0	0

IV. CONCLUSION

The application of Bernoulli polynomial to numerical solution of differential equations of second order and its properties are significant to cut down the differential equations to system of algebraic equations. Illustrations are given to check the efficiency, accuracy and validity of the proposed method and analyzed with one of the existing method Rationalized haar function (RHF).

REFERENCES

[1] Baggiano, Anthony Lincoln (1963). Numerical Solution to differential equations using Legendre Polynomials. Master's Thesis, Agricultural and Mechanical college of Texas. Texas A&M University

[2] Morris, A., & Horner, T. (1977). Chebyshev Polynomials in the numerical solution of differential equations. Mathematics of Computation, 31(140), 881-891. Doi:10.2307/2006119.

[3] L.C. Eisenhart and G. E. Trapp, Computation of Hermite Polynomials, Mathematics of Computation, 27(123), 1973.

[4] Z. Kalateh Bojdi , S. Ahmadi-Asl And A. Aminataei, Operational matrices with respect to hermite polynomials and their applications in solving linear differential equations with variable coefficients, Journal of linear and topological algebra, 02(02), 2013, 91- 103.

[5] G. Arfken, Mathematical Methods for physicists, Academic Press, San Diego (1985).

[6] F. Costabile, F. Dellaccio, M. I. Gualtieri, A New approach to Bernoulli Polynomials, Rendiconti Di Matematica, Serie Vii, 26, 2006, 1-12.

[7] H. S. Ramane, S. C. Shiralashetti, R. A. Mundewadi, R.B. Jummannaver, Numerical solution of fredholm integral equations using hosoya polynomial of path graphs, American Journal of Numerical Analysis, 5(1), (2017), 11-15, Doi: 10.12691/Ajna-5-1-2.

[8] S.C. Shiralashetti, H.S. Ramane, R. A. Mundewadi, R.B. Jummannaver, A comparative study on haar wavelet and hosaya polynomial for the numerical solution of fredholm integral equations, Applied Mathematics and Nonlinear Sciences, 3(2) (2018), 447–458.

[9] Raju B. Jummannaver, Ivan Gutman, R. A. Mundewadi, On zagreb indices and coincides of cluster graphs, Bulletin of the international mathematical virtual institute, 8, 2018, 477-485.

[10] R. A. Mundewadi, H. S. Ramane, R. B. Jummannaver, Numerical solution of first order delay differential equations using hosoya polynomial method, Indian J. Discrete Math., 4(1) ,2018, 1 – 11.

[11] F. Harary, Graph Theory, Addison Wesley, Reading, 1968.

[12] H. S. Ramane, R. B. Jummannaver, Note on the forgotten topological index of molecular graphs in drugs, Applied mathematics and nonlinear science, 1(2), 2016, 369-374.

[13] H. S. Ramane, K. S. Pise, D.D. Patil, Note on inverse sum indeg index of graphs, AKCE International Journal of Graphs and Combinatorics, DOI: 10.1016/j.akcej.2019.12.002.

[14] S. C. Shiralashetti, R. A. Mundewadi, Modified wavelet full-approximation scheme for the numerical solution of nonlinear volterra integral and integro-differential equations, Applied mathematics and nonlinear sciences, 1(2), 2016, 529-546.

[15] S. C. Shiralashetti, R. A. Mundewadi, Bernoulli wavelet based numerical method for solving fredholm integral equations of the second kind, Journal of information and computing sciences, 11(2), 2016, 111-11.

[16] R. A. Mundewadi, B. A. Mundewadi, Hermite wavelet collocation method for the numerical solution of integral and integro-differential equations, International journal of mathematics trends and technology, 53(3), 2018, 215-231.

[17] M. Razzaghi, Y. Ordokhani, Solution of differential equations via rationalized haar functions, Mathematics and computers in simulation, 56, 2001, 235–246.

[18] S. Bazm, Bernoulli polynomials for the numerical solution of some classes of linear and nonlinear integral equations, Journal of computational and applied mathematics, 275, 2015, 44-60..